Parrondo's paradox and the Brownian ratchet

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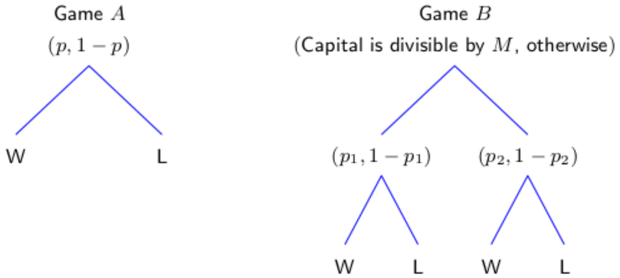
Motivation and objective

- Is there any way of combining two losing strategies to form a winning formula?
  e.g: trueling problem, costly signalling, interplay of redundancy and pleiotropy.
- Can a drift or order be generated from randomness?
  e.g: the ratchet and pawl machine.
- If possible, is the formula valid for all range of values of the variables for a given system or does the domain have restriction?

Parrondo's games

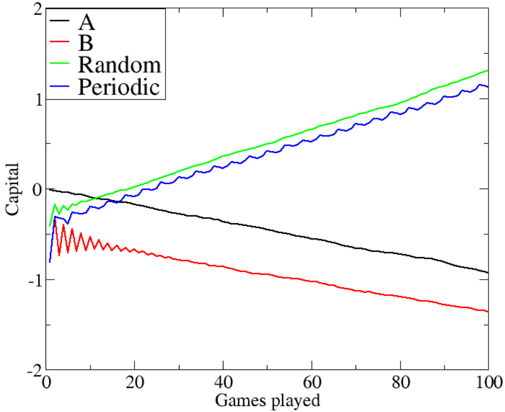
The rules of evolution

- We start with 0 capital($X_0 = 0$).
- If we win, $X_{n+1} = X_n + 1$
- If we lose, $X_{n+1} = X_n - 1$



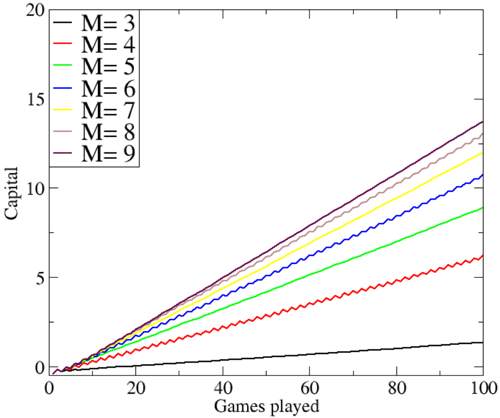
The original game

- The winning probabilities:
  $p^A = \frac{1}{2} - \epsilon, \;\; p^B_1 = \frac{1}{10} - \epsilon, \;\; p^B_2 = \frac{3}{4} - \epsilon, \textit{ where } \epsilon = 0.005$



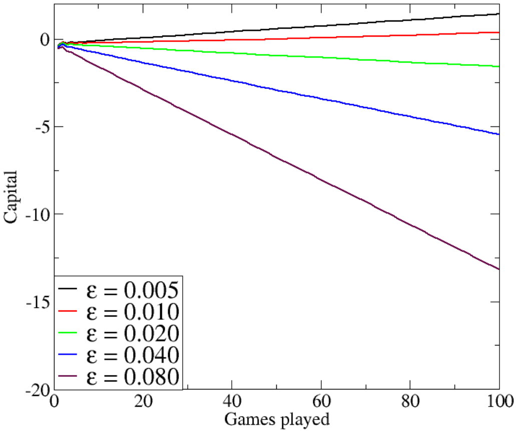
Variation with M

- As M increases, coin 2 is played more often, leading to a more winning situation. Here $\epsilon = 0.005$.



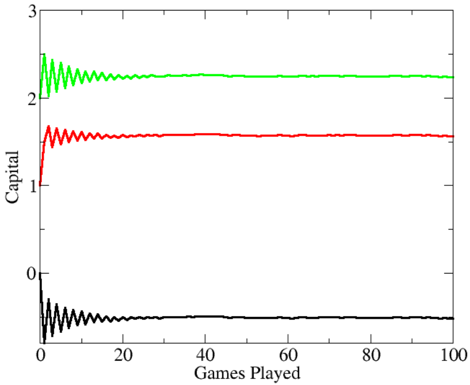
Variation with ϵ

- The variation of ϵ in multiplicative steps. Here, $M = 3$.



Fairness of game B

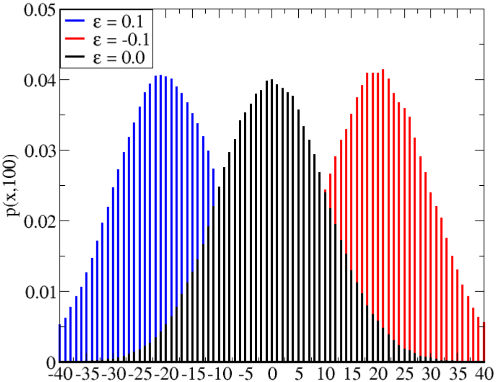
- The expectation value of the capital,
  $E[X_{n+1}|X_0, X_1, ..., X_n] = X_n$
  where $n \in Z_+$ and X_n is the capital after the n_{th} game.



Probability distribution function
game A

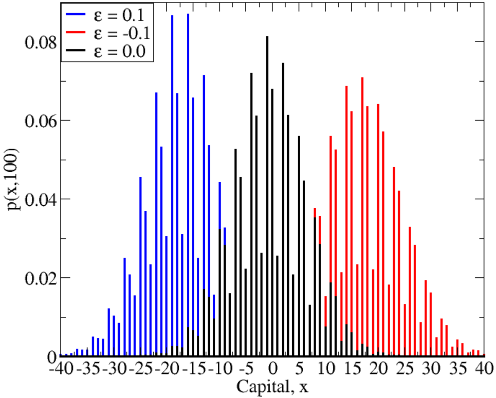
- we define the central probability $\hat{p}(x, n)$,as follows:

$$\hat{p}(x, n) = \frac{p(x,n+1) \ + \ 2p(x,n) \ + \ p(x,n-1)}{4}$$



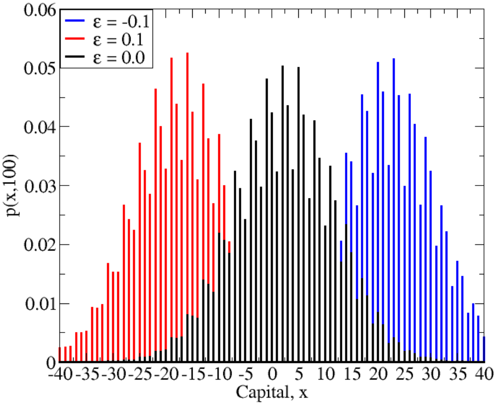
Probability distribution function
game B

- The PDFs are plotted for $M = 3$, with the previously mentioned values of probabilities.



Probability distribution function
randomized games

- The games are randomly mixed to obtain the PDFs.

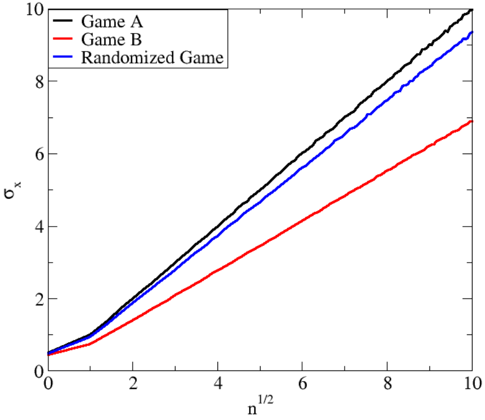


Variation of σ_x with n

- The normal distribution of game A is such that, $\mathcal{N}(n(p-q), 4npq)$, where $q = 1 - p$.
- For $p = \frac{1}{2} - \epsilon$, $\langle x \rangle = -2n\epsilon$ and $\sigma_x = 2\sqrt{npq}$.
- $\sigma_x \sim \sqrt{n}$
  $\Rightarrow$ the more compact the values, the less would be the slope of σ_x *vs* $\sqrt{n}$ curve.

Behavior of the games

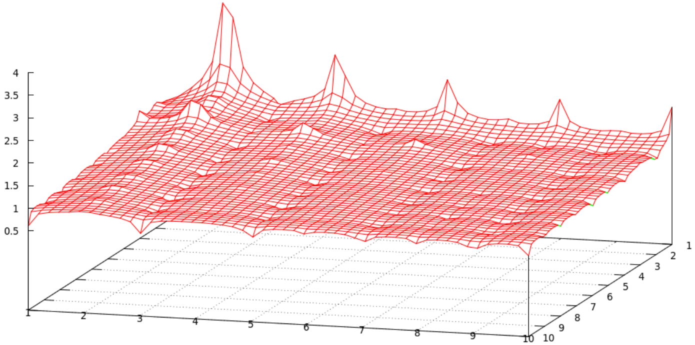
- The compactness of the PDFs are compared with respect to this by plotting σ_x vs n.



Variation with mixing sequence $[a, b]$

$M = 3$

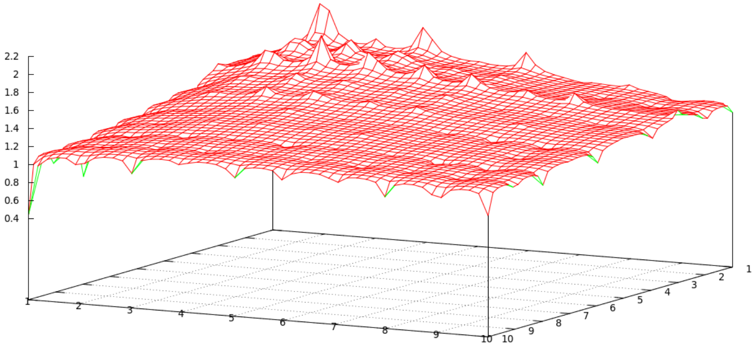
- Plot of X_n w.r.t mixing sequence $[a, b]$.
- Here, $M = 3$, $p = 0.5$, $p_1 = 0.1$, $p_2 = 0.75$, $n = 100$.



Variation with mixing sequence $[a, b]$

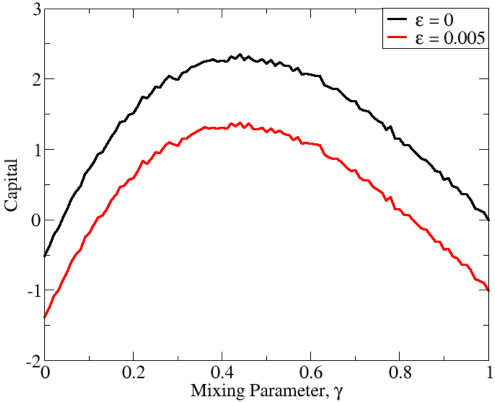
$M = 5$

- Here, $M = 5, \ p = 0.5, \ p_1 = 0.1, \ p_2 = 0.634.$



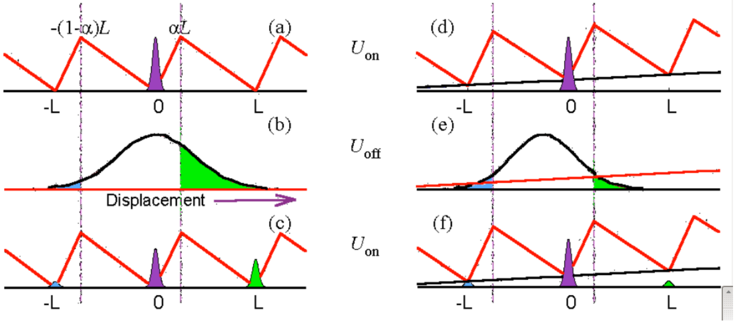
Variation with mixing parameter γ

- The probability of choosing game A at random is associated with the mixing parameter γ, where $0 < \gamma < 1$.



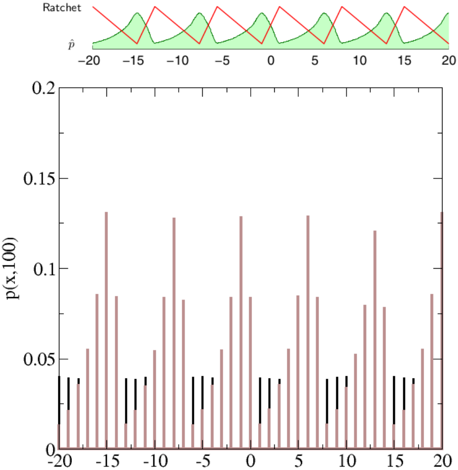
The ratchet potential

- Brownian particle in 1D.
- Asymmetric sawtooth potential is *flashed* on and off.



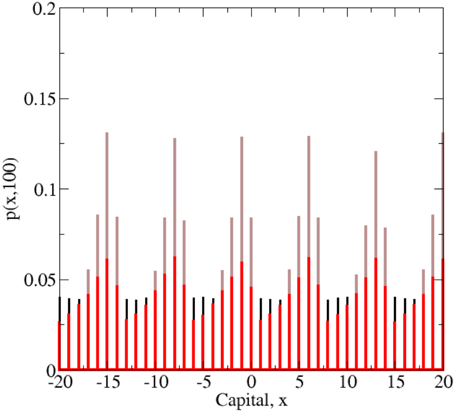
Superposition of PDFs

- $M = 7,\ p = 0.5,\ p_1 = 0.075,\ p_2 = 0.6032.$



Breaking the stationary pattern

- The flashing of game A, leads to the breakdown of the probability distribution and sets the drift.



Analogy with Brownian ratchet

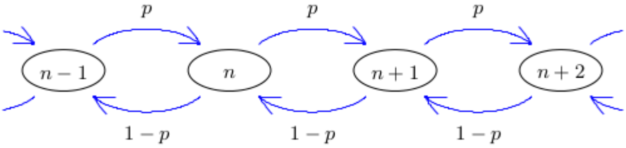
- The one to one correspondance between different quantities of the continuum and discrete models.

Table 1. The relationship between quantities in Parrondo's games and the Brownian ratchet.

Quantity	Brownian Ratchet	Parrondo's Paradox
Source of Potential	Electrostatic, Gravity	Rules of games
Switching	U_{on} and U_{off} applied	Games A and B played
Switching Durations	for τ_{on} and τ_{off}	a and b
Duration	Time	Number of games played
Biasing	Macroscopic field gradient	Parameter ϵ
Transport Quantity	Brownian particles	Capital
Measurable Output	Displacement x	Capital amount X_n
External Energy	Switching U_{on} and U_{off}	None
Potential Shape	Depends on α	Probabilities p_1, p_2 and M
Mode of Analysis	Fokker-Planck equation	Discrete-time Markov chain

Discrete time Markov chain

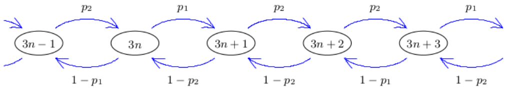
game A



The transition matrix $\mathbb{P}_A$:

$$\mathbb{P}_A = \begin{pmatrix} 0 & 1-p & \cdots & \cdots & (p) \\ p & 0 & 1-p & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & p & 0 & 1-p \\ (1-p) & \cdots & \cdots & p & 0 \end{pmatrix}$$

Discrete time Markov chain
game B

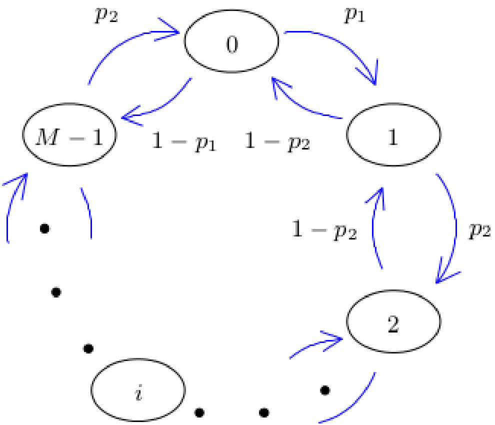


The transition matrix $\mathbb{P}_B$:

$$\mathbb{P}_B = \begin{pmatrix} 0 & 1-p_2 & & & & & (p_2) \\ p_1 & 0 & \ddots & & & & \\ & p_2 & \ddots & 1-p_2 & & & \\ & & \ddots & 0 & 1-p_1 & & \\ & & & p_2 & 0 & 1-p_2 & \\ & & & & p_1 & 0 & \ddots \\ (1-p_1) & & & & & \ddots & \ddots \end{pmatrix}$$

The modulo M game

- $Y_n \equiv X_n mod M$



Randomized games

- The game A is chosen with probability γ.
- $q_1 = \gamma p + (1-\gamma)p_1$, when capital is a multiple of M
- $q_2 = \gamma p + (1-\gamma)p_2$, otherwise
- $\mathbb{P}^R$ is attained by replacing p_1 by q_1 and p_2 by q_2.

Analytic results

- $\pi(n) \rightarrow$ *State of capital after n attempts*
- $\pi(n) = \mathbb{P}^n \pi(0)$
- $\pi(0) = [\cdots, 0, 1, 0, \cdots]^T$
- For the mixing of games via $[a, b]$

$$\pi(n)^{[a,b]} = \mathbb{P}_X^n \pi(0), \text{ where}$$

$$\mathbb{P}_X = \begin{cases} \mathbb{P}_A & \text{if } (n-1) mod(a+b) \; < a, \\ \mathbb{P}_B & \text{otherwise} \end{cases}$$

$$\text{where, } n = 1, 2, \cdots$$

- For the mixing of games via γ

$$\pi(n)^{\gamma} = \mathbb{P}_R^n \pi(0) \text{ where, } \mathbb{P}_R = \gamma \mathbb{P}_A + (1-\gamma)\mathbb{P}_B$$

Calculation of statistical quantities

- $\mathbf{x} = [-N, \cdots, N]$
- $\mu_n = \mathbf{x}\pi(n)$
- $\sigma_n = \sqrt{(\mathbf{x} - \mu_n)^2 \pi(n)}$
- The stationary state is reached when,

$$\lim_{n \to \infty} \pi(n) = \pi$$

- In stationary state, the equation and solution are:

$$(\mathbb{I} - \mathbb{P})\pi = 0$$

$$\pi = \frac{1}{D} \text{diag}(\text{cofac}(\mathbb{I} - \mathbb{P}))$$

Stationary state solutions

- For M=3,

$$\pi^B = \frac{1}{D}\begin{bmatrix} 1 - p_2 + p_2^2 \\ 1 - p_2 + p_1 p_2 \\ 1 - p_1 + p_1 p_2 \end{bmatrix}$$

  where, $D = 3 - p_1 - 2p_2 + 2p_1 p_2 + p_2^2$
- For game A, $p_1 = p_2 = p$,then

$$\pi^A = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- For game B with no bias($\epsilon = 0$),we have

$$\pi^B = \frac{1}{13}\begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$$

Constraints of the game

$$p_{win} = \sum_{j=0}^{M-1} \pi_j p_j$$

- For the game A,

$$\frac{1-p}{p} > 1$$

- For the game B

$$p_{win}^B = \pi_0 p_1 + (1 - \pi_0) p_2$$

$$\frac{(1-p_1)(1-p_2)^2}{p_1 p_2^2} > 1$$

- For the randomized game

$$\frac{(1-q_1)(1-q_2)^2}{q_1 q_2^2} < 1$$

Role of asymmetry and randomness
few examples

- The interplay of asymmetry and randomness leads to a drift.

Scenario	Source of Randomness	Asymmetry
Brazil nut paradox	Shaking the container	Particle sizes/Field
Longshore drift	Waves breaking on the beach	Geometry/Friction
Restaurant check	Waiter's error rate	Information
Buy-low, sell-high	Market fluctuations	Price
2-Girl paradox	Bill's arrival times	Train phase (time)

Summary

- The analogy between *Parrondo's games* and *Brownian flashing ratchet* are observed. The flashing of games leads to drift of capital towards a positive value.
- Can serve as a discrete-continuum interface model.
- Asymmetry and randomness in a system can lead to a Parrondonian effect.
- The effect is not a Paradox and is restricted to small volume of phase space.
- Can be useful in the context of evolutionary biology, social dynamics, stock market analysis etc.

New outlook

- History dependant, multiplayer Parrondo's games are devised and studied.
- Allison mixture $\Rightarrow$ Sequencing in codes of DNA.
- To find more examples which map to physical processes.

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