Parrondo's paradox and the Brownian ratchet

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Contents

1 Introduction

The concept of opting for a losing move to win in the long run is a well-used and maybe cliched strategy, but the Physics of it is still anew. In almost every domain of human life (or beyond) these effects are seen. Be it the game of chess, where sacrifising a less powerfull member properly may lead to a winning situation after a few moves. Be it the concept of counter attack in the game of football, where one draws out the opponent first(losing move), to win in the following moves. Although the feel of these examples have a sence of unity, the study of such complex games where a large number of parameters are present is tideous, if not impossible.

The probabilistic games proposed by Juan M. Parrondo serves as an wonderful yet simple model of such an event: Winning by the combination of losing games. We intend to study the games numerically as well as analytically and try to see if it is indeed a Paradox or not.

A similar model on the continuum domain is that of a Brownian ratchet. We take the case of a *flashing ratchet* where an asymmetric saw-tooth potential is flashed randomly to affect the motion a Brownian particle in 1D. Here the particle shows a drift towards one of the sides and even if we add a biasness to the opposite end, upto a certain range of biasness, the initial direction of drift is maintained.

We would also like to see, if this "Parrondonian effects" are seen in other context, in nature. If so, we would try to see what is the cause of such a conuter-intuitive behavior.

2 Parrondo's Paradox

The process of combining two losing games to arrive at a winning situation, is demonstrated numerically with the example of a coin tossing game. The rules are as follows:

2.1 Game A

A biased coin is tossed that has a probability (p) of winning. It is set as less than $\frac{1}{2}$, making it a losing game. We define a positive parameter ϵ to introduce bias, i.e.

$$p = \tfrac{1}{2} - \epsilon.$$

2.2 Game B

Consisting of 2 biased coins such that,

$$p_1 = \tfrac{1}{10} - \epsilon$$
$$\text{and}$$
$$p_2 = \tfrac{3}{4} - \epsilon.$$

Coin 2 is played if the capital of the player is a multiple of an integer M(say, 3). Otherwise, coin 3 is played. So, the coin 3 is tossed a little bit more times on the average, than the coin 2. But the probabilities are such that on an average

B is a losing game. If value of M is changed, the weightages would also change and the probabilities may have to reassgined new values to make the game B a losing one.

2.3 Simulation

We start, initially, with 0 money. If the player wins, unit amount of money is added to his capital. If loses, unit amount is subtracted.
We chose $\epsilon = 0.005$. We played the games 100 times in the 4 following ways:
1. Game A only.
2. Game B only.
3. Game A and B randomly.
4. Game A and B, played in the sequence AABB.

An average over 50,000 configurations were done to obtain the variation of capital w.r.t the number of games played. The plot is shown in Fig.-(1).

2.4 Plot of capital vs games played

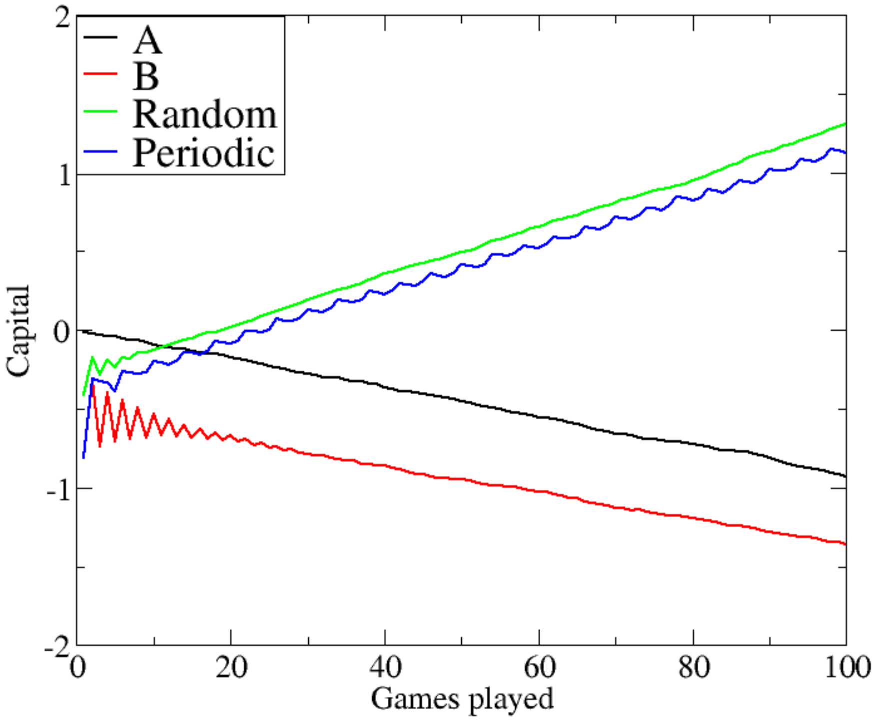

Figure 1: The plot show the variation of capital w.r.t the number of times the games are being played. While the games A (≈ -0.93) and B (≈ -1.36) are individually losing ones, when played together in a periodic (≈ 1.12) or randomised (≈ 1.31) fashion, they lead to a winning combination. The numbers within parentheses indicate the approximate values of the capital after the 100 games.

2.5 Variation with M

The parameter M decides whether the coin 2 or coin 3 is played more often. As M grows, coin 3 is played more and more number of times in the long run, giving the game B a relatively winning edge. The expected increase in capital with increase of M is seen in Fig.-2. Here the value of $\epsilon = 0.005$, throughout.

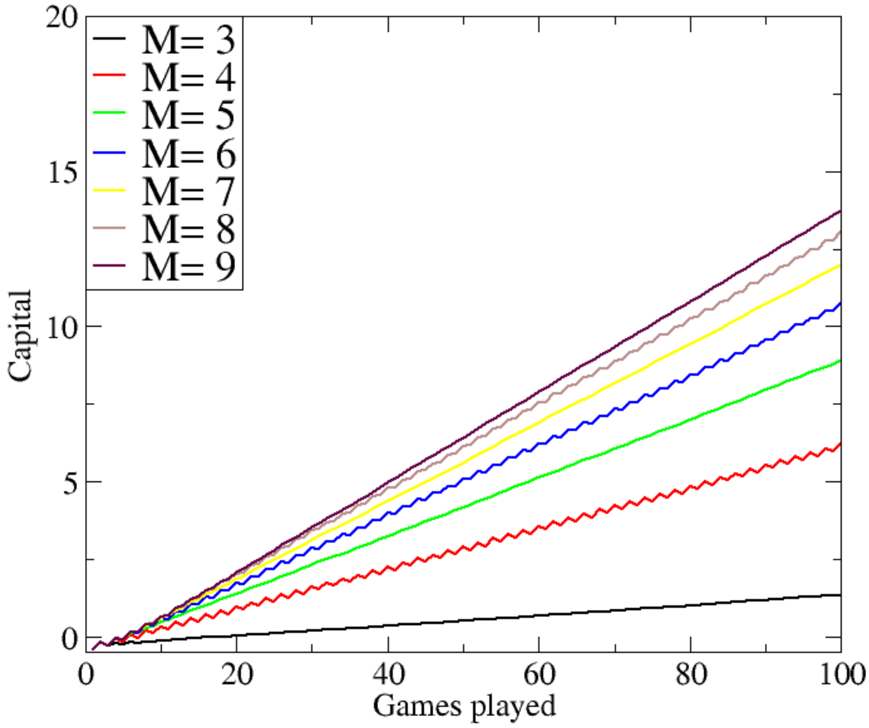

Figure 2: The plot show the variation of capital w.r.t the number of times the games are being played. Here we have plotted only the random sequence of A and B. Different curves show the variation when M is varied from 3 to 9. It is seen that the variation is step wise in case of even M and continuous, otherwise. The value of capital though increases with increase in M, the rate of that slows down gradually. Here, $\epsilon = 0.005$, for all the cases.

2.6 Variation with ϵ

The parameter ϵ decides the winning(or losing) probabilities of the games. As ϵ grows, the winning probabilities decrease. The expected decrease in capital with increase of ϵ is seen in Fig.-3. The value of M is kept constant. Here, we took $M = 3$.

2.7 Fairness of the game B

The fairness of a game can be defined in terms of the drift of the capital or the expectation value of the capital after a number of trials. If a game is fair, there should be no drift and hence the expectation value of capital after $(n+1)_{th}$ turn, would be equal to the value of capital attained after the n_{th} turn, i.e;

$$E[X_{n+1}|X_0, X_1, ..., X_n] = X_n$$

where $n \in Z_+$ and X_n is the capital after the n_{th} game.

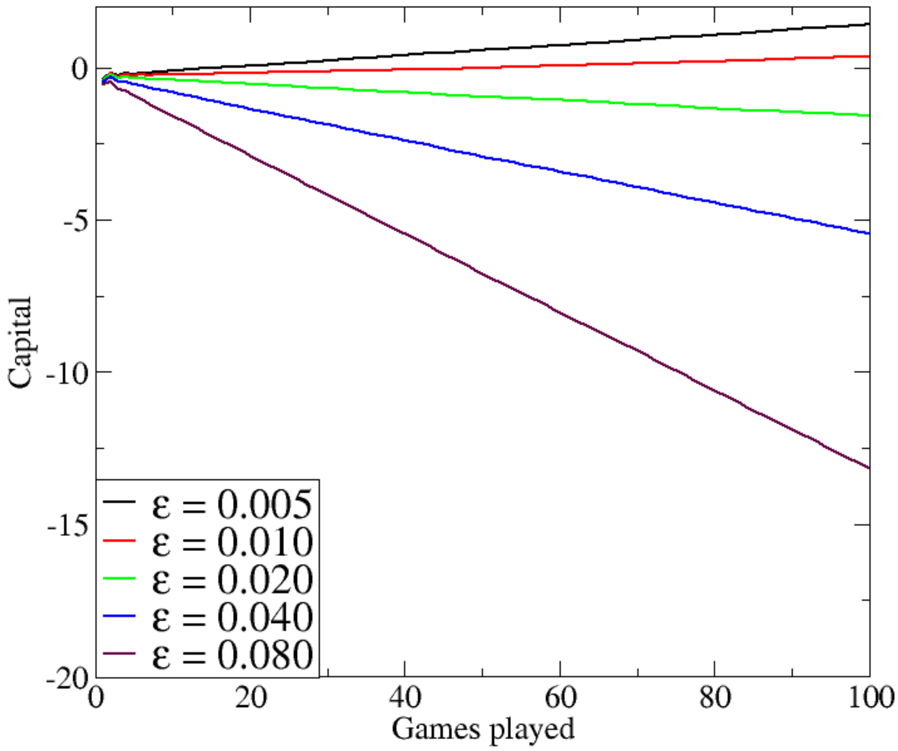

Figure 3: The plot show the variation of capital w.r.t the number of times the games are being played. Here we have plotted only the random sequence of A and B. Different curves show the variation when ϵ is varied from 0.005 to 0.080 in multiplicative steps. Here, $M = 3$. We see the games go from a winning combination to a losing one as ϵ is slowly increased.

Without the introduction of biasing parameter ϵ, the game A is fair by construction, ($p_{win} = \frac{1}{2}$). The nature of game B is somwhat complex as it is capital dependant, implying $E[X_1|X_0] < X_0$ when X_0 is a multiple of M, and $E[X_1|X_0] > X_0$ when it is not a multiple of M.

Thus to know the nature of game B, we try it out a number of times and see the averaged value of capital in that time frame, for the three non-redundant initial value capitals, viz. $0, 1, 2$. The plot is shown in Fig.-4.

2.8 Distribution and behaviour of the games

Although the average value of capital tends to a steady value after a large number of attempts, these, however does not give us the entire information about the games or the capital. As the capital may vary between $[-n, n]$ after the game is being played n times, the probability distribution function of capital does say how the capital is distributed about the mean, i.e, how far-off can the games be from a fair condition and what is the amount of deviation. In order

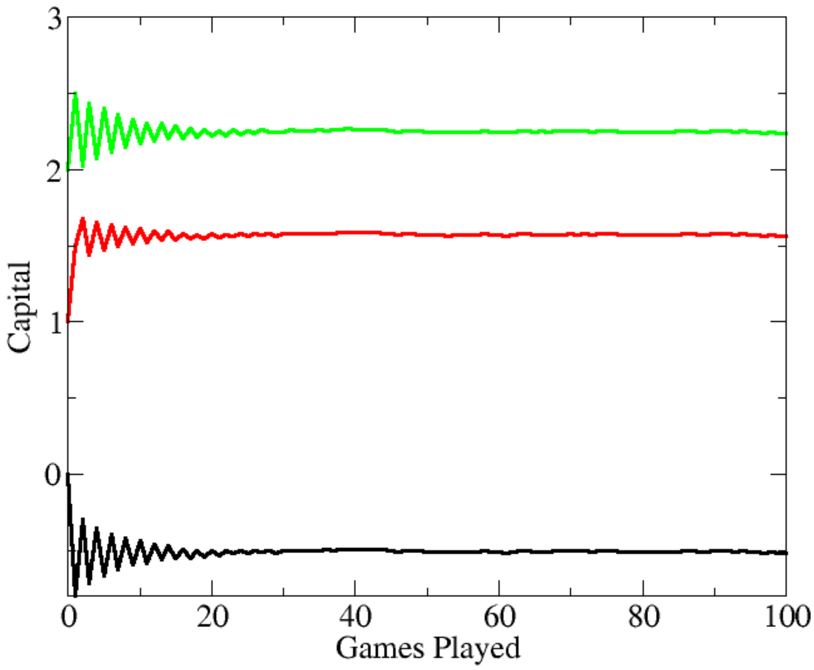

Figure 4: The plot show the variation of capital w.r.t the number of times the games are being played(Here,$n = 100, n_{configurations} = 50000$). Here we have plotted only the game B. Different curves show the variation when initial capital is $0, 1, 2$. Here, $M = 3$, $\epsilon = 0$. The games initially pick up a winning or losing trend depending on the initial capital. But soon, settles down towards a driftless steady value,indicating fairness of game B.

to capture this, we define the central probability $\hat{p}(x, n)$,as follows:

$$\hat{p}(x, n) = \frac{p(x,n+1)\ +\ 2p(x,n)\ +\ p(x,n-1)}{4}$$

We plot it for games A, B and randomized games for 3 different values of the biasing parameter, $\epsilon = -0.1, 0\ and\ 0.1$. They are shown in figures 5.

2.9 Standard deviations of the games

The game A has a Gaussian PDF, which can be taken as a benchmark of judging the "behaviour" of games with number of times they are played. The normal distribution of game A has the following parameters, $\mathcal{N}(n(p - q), 4npq)$, where

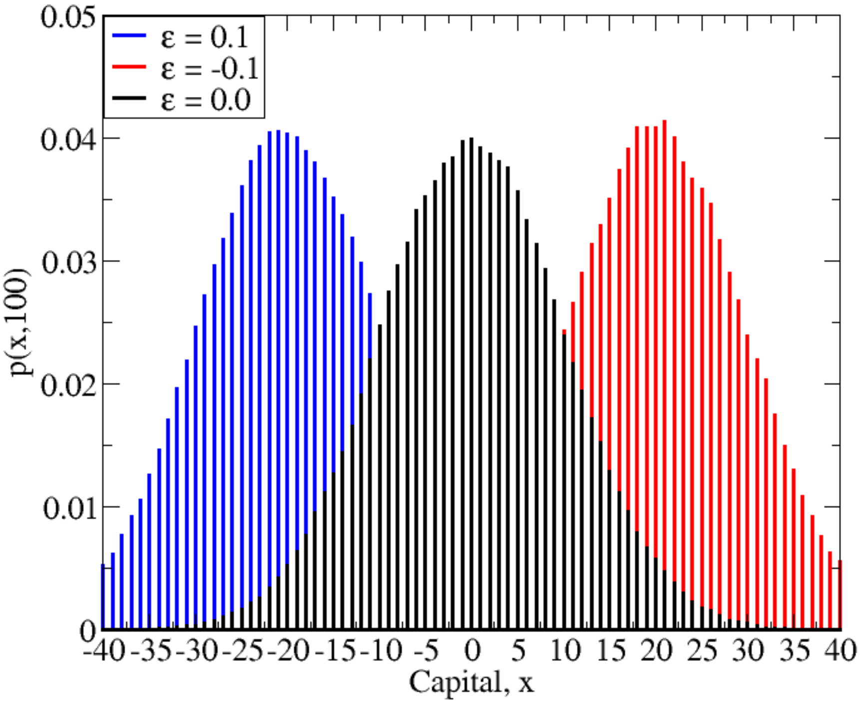

Figure 5: The plot shows the distribution of capital, x after n_{th} game being played. Here we have plotted only game A. Different curves show the variation when $\epsilon = -0.1, 0$ *and* 0.1. The PDF is a Gaussian which becomes flatter and flatter as n increases. The peak corresponds to the mean value of capital, which is drifted towards positive value when $\epsilon = -0.1(winning)$ and towards a negative value when $\epsilon = 0.1(losing)$.

$q = 1 - p$ is the losing probability.

For $p = \frac{1}{2} - \epsilon$, the mean value of capital, after n_{th} game is $\langle x \rangle = -2n\epsilon$ and the standard deviation, $\sigma_x = 2\sqrt{npq}$.

The linear dependancy of σ_x with $\sqrt{n}$ says the more compact the values are, the less would be the slope of σ_x vs $\sqrt{n}$ curve, implying the well behaviour of the game.

So, in the figure 8, we plot the variation of σ_x w.r.t $\sqrt{n}$ for the games A, B, and randomized games. We compare the slopes to see the behaviour of games.

2.10 Mixing sequences

The next question that arises is how far do the games have to mix to break the pattern of game B. Here the broken pattern will be manifested by a positive

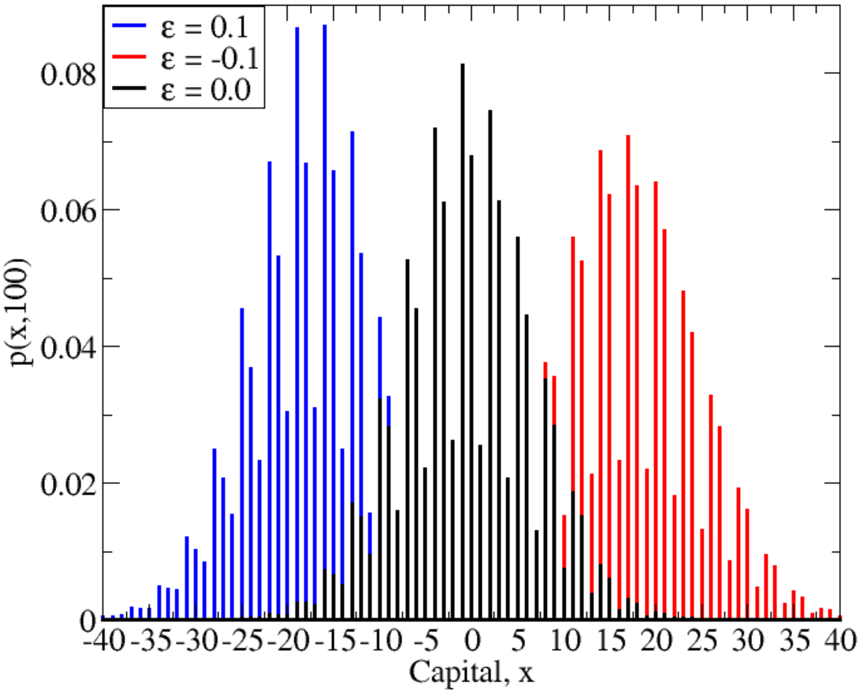

Figure 6: The plot shows the distribution of capital, x after n_{th} game being played. Here we have plotted only game B. Different curves show the variation when $\epsilon = -0.1, 0$ *and* 0.1. The PDF is jagged which becomes flatter and flatter as n increases. The peak is drifted towards positive value when $\epsilon = -0.1(winning)$ and towards a negative value when $\epsilon = 0.1(losing)$.

value of capital. We play a total of 100 games, in different mixing sequences $[a, b]$, where the notation refers to playing the game A a times, followed by playing the game B b times and so on.

The variation of capital is plotted w.r.t a and b, in figure 9. The plot shows that when the games are repeatded at large interval, there is hardly any gain. Whereas, if the games are mixed more and more frequently, the gain increase. The result suggests that the mixing of the game is the key to producing a gain.

2.11 Randomized mixing via the parameter γ

So far, we have played the randomized version of the games with equal probability of choosing between A and B. This can be modified with a parameter γ ($0 \leqslant \gamma \leqslant 1$) that determines the probability of choosing game A. The variation of capital w.r.t γ is shown in Fig.-10.

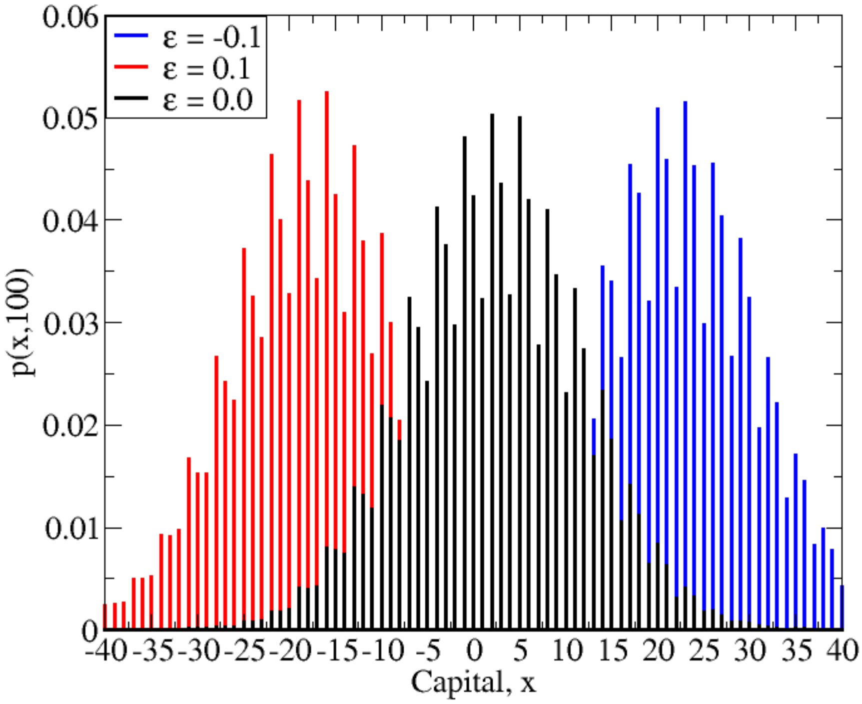

Figure 7: The plot shows the distribution of capital, x after n_{th} game being played. Here we have plotted only the randomized games. Different curves show the variation when $\epsilon = -0.1, 0$ *and* 0.1. The PDF is less jagged than that of game B which becomes flatter and flatter as n increases. The peak is drifted towards positive value when $\epsilon = -0.1(winning)$ and towards a negative value when $\epsilon = 0.1(losing)$.

3 Brownian ratchet

3.1 Flashing ratchet in 1D

We consider a Brownian particle in 1D. An asymmetric sawtooth potential is *flashed* on and off, to affect the motion of the particle. Without any other biasness, the asymmetry and randomness, together, sets the particle in a drift towards one of the sides. Even if a bias (a small gradient) is applied to the opposite end to balance this, upto a value of biasnessness, the direction of initial drift is maintained as shown in Fig. -11.

3.2 Explanation of the games

The games were designed as an illustration of Brownian ratchet problem. It is a system where brownian noise is converted into drifting motion in particular direction using a "ratchet" potential. The analogy between the Parrondo's games and Brownian ratchet helps in understanding the later through the former.

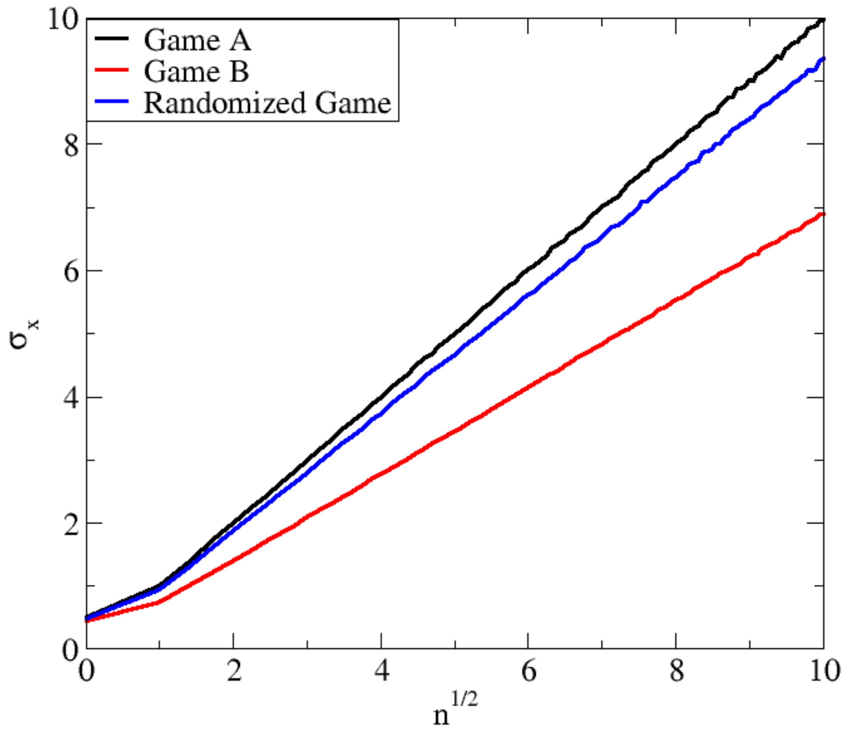

Figure 8: The standard deviation of the capital after the n_{th} game $(\sigma_x(n))$ is plotted against $\sqrt{n}$. Although the PDF of game B was more jagged, but the slope is less for that case, indicating that game B is atleast as well behaved as(if not more) game A. The slope for the randomized game's curve is intermediate, showing a moderate compactness of values. It is as if the pattern of game B is broken by the game A to arrive at the randomized case. Here, the value of $\epsilon = 0$ and the averaging is done over $100, 100$ sample paths.

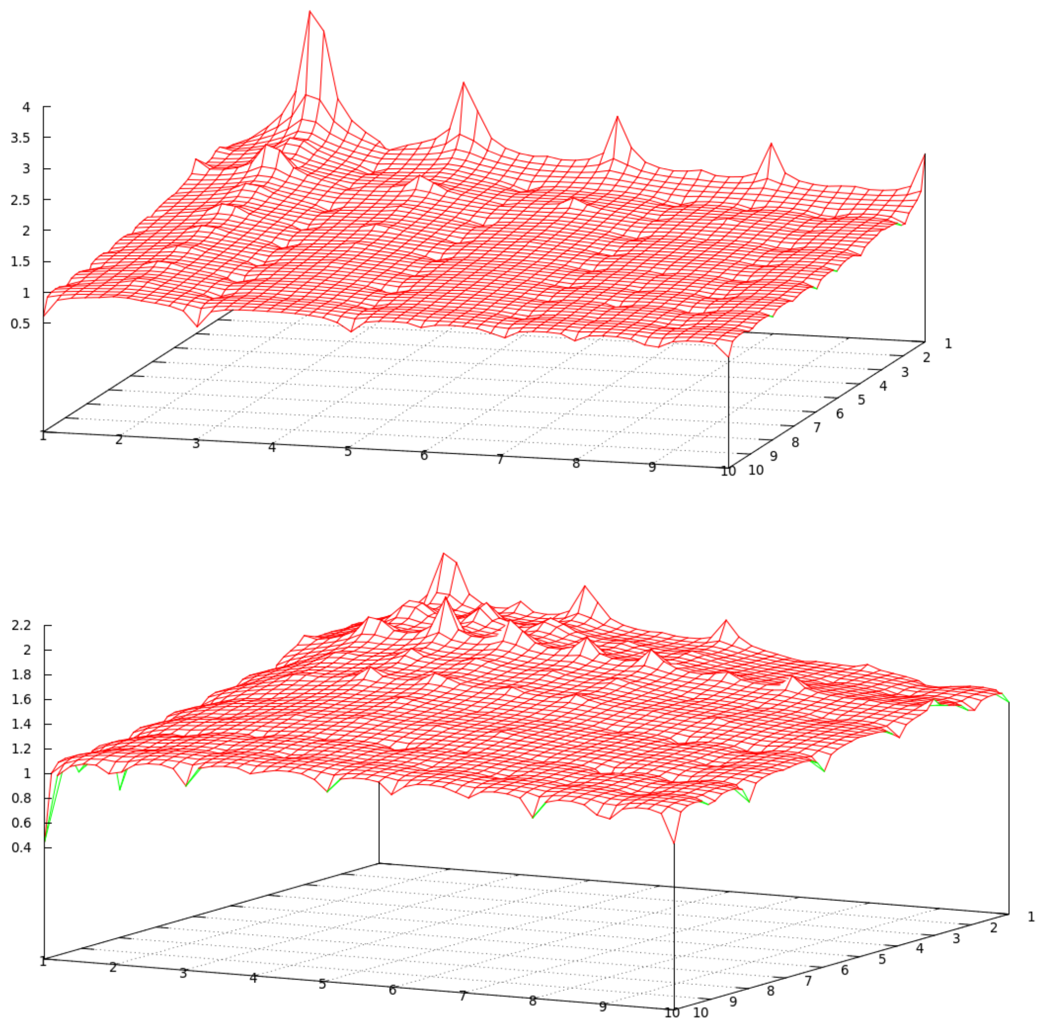

Figure 9: The plots show the variation of capital after n_{th} game, for different deterministic mixing sequences between the games. Here, $n = 100$ and mixing sequence is used as $[a, b]$. For the 1^{st} diagram, $M = 3,\ p = 0.5,\ p_1 = 0.1,\ p_2 = 0.75$, for the 2^{nd} diagram, $M = 5,\ p = 0.5,\ p_1 = 0.1,\ p_2 = 0.634$.

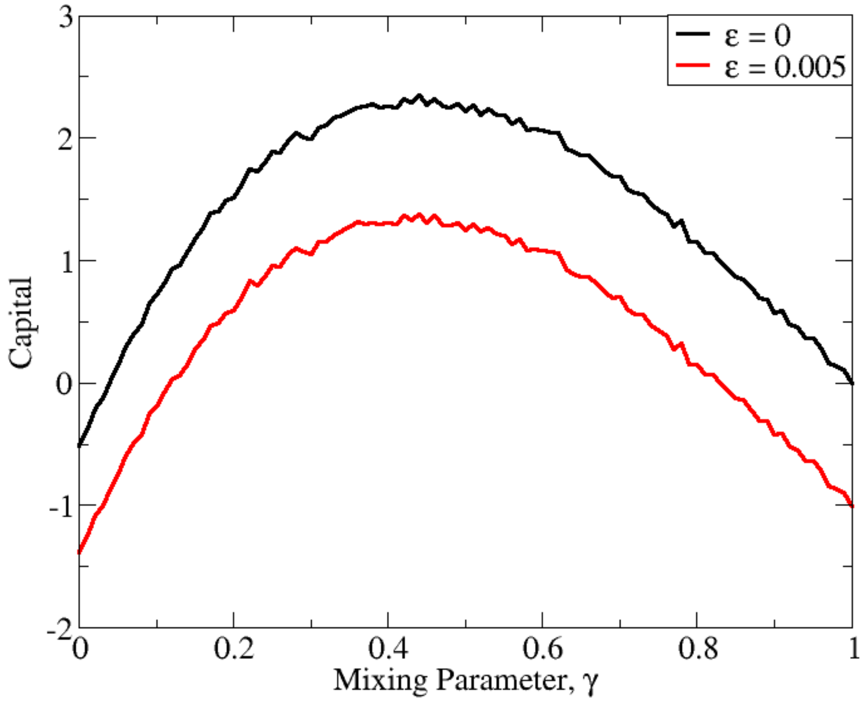

Figure 10: For the biased game, when $\gamma = 0$(*only game B*) or $\gamma = 1$(*only game A*), the capital drifts towards a losing value. But in the intermediate region, there is a portion of the curve that has a positive average capital. The optimal value of γ for which the mixing produces maximum gain can also be found from such diagrams. Here, we have considered 100 games, averaged over 50,000 configurations. The plots are for $\epsilon = 0$ (*top*) *and* $\epsilon = 0.005$ (*bottom*)
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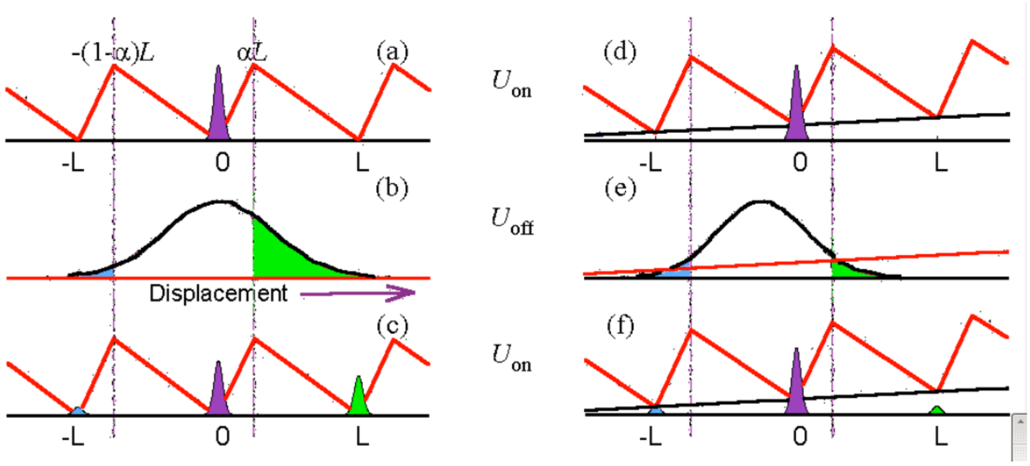

Figure 11: The asymmetry kicks in through the value of α. Here, $\alpha < 1/2$. The Initial drift occurs towards the steeper slope. Even after a slight tilting of the potentials, this direction is maintained. $U_{on} \equiv$ *Asysmmetric sawtooth potential* and $U_{off} \equiv$ *flat potential*.

3.3 PDF in ratchet potential

The flat potential($V(x) = 0$) generates a Gaussian PDF, which becomes more and more flat with passing time. If there is a slope to it, the PDF additionally shows a drift downhill. For an asymmetric saw-tooth like potential, the Gaussian PDF gets distortedwithin the region in the following way: Here the PDF of the capital shows a similar nature for the games A(*flat potential*) and game B(*asymmetric saw-tooth*). The periodic nature is captured via different initial values of the capital.

3.4 Analogous quantities

Although the Brownian ratchet is a continuous temporal and spatial problem, there are certain analogies which can be drawn between these 2. As the Parrondo's games are much more mathematically tractable, hence they can be useful in analysis of the former. The list of analogous quantities are mentioned below:

3.5 Breaking of the equilibrium distribution

The PDF due to game B was similar to that of PDF of a ratchet potential, in equilibrium. The pattern of PDF in equilibrium is broken by momentary switching on and off of the game A. In a sense, the PDF which was distributed about the minima of a sawtooth potential, is spread out by the onset of game A (*flat potential*). The probability density that builds up near the steep edge,

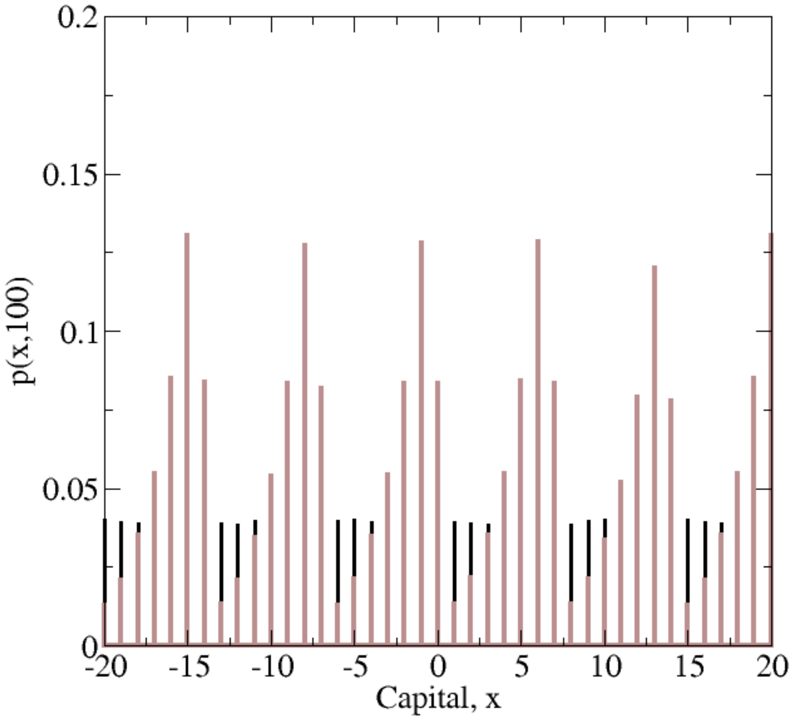

Figure 12: The PDFs of games A(*in black*) and game B(*in brown*) are superimposed on the same plot here. The parameters are, $M = 7,\ p = 0.5,\ p_1 = 0.075,\ p_2 = 0.6032$.

Table 1. The relationship between quantities in Parrondo's games and the Brownian ratchet.

Quantity	**Brownian Ratchet**	**Parrondo's Paradox**
Source of Potential	Electrostatic, Gravity	Rules of games
Switching	U_{on} and U_{off} applied	Games A and B played
Switching Durations	for τ_{on} and τ_{off}	a and b
Duration	Time	Number of games played
Biasing	Macroscopic field gradient	Parameter ϵ
Transport Quantity	Brownian particles	Capital
Measurable Output	Displacement x	Capital amount X_n
External Energy	Switching U_{on} and U_{off}	None
Potential Shape	Depends on α	Probabilities p_1, p_2 and M
Mode of Analysis	Fokker-Planck equation	Discrete-time Markov chain

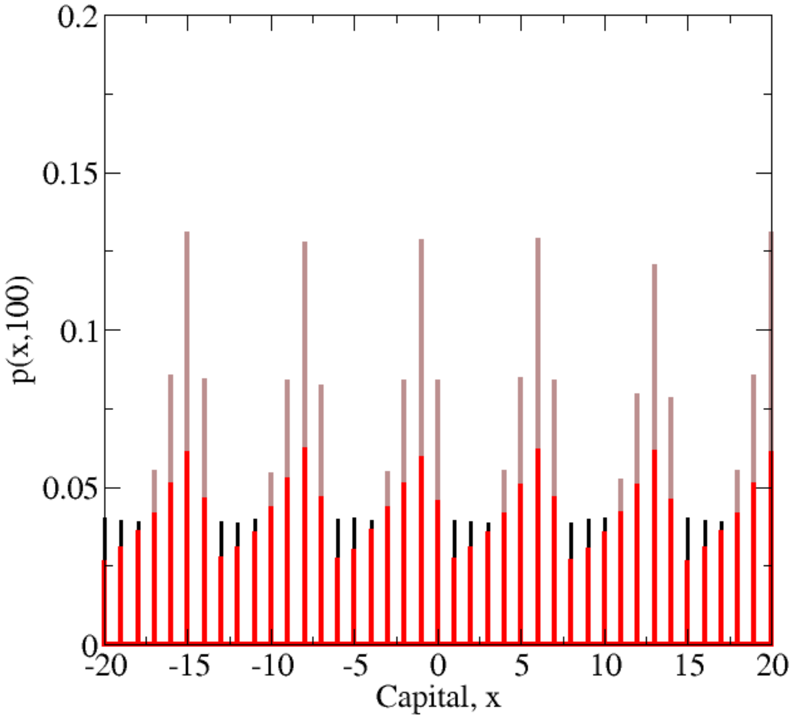

Figure 13: The plots of PDFs of games A, B and randomized ones are superimposed. The jagged pattern of game B is broken by game A to form a relatively smoother pattern, which allows for drift of capital.

during the presence of game A, allows for a probability current towards the next minima. This leads to the breakdown of the equilibrium probability distribution and sets the drift.

4 Analytic results

We want to use the Discrete-time Markov chains(henceforth, called as "DTMC") to analysis the games. A "modulo game" would also be defined to attain steady state solutions.

4.1 Game A

For the game A, the DTMC is represented in Fig.- 14.

The transition matrix $\mathbb{P}_A$ is given as follows:

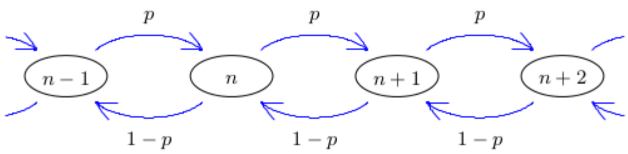

Figure 14: A part of the "doubly-infinite" DTMC is shown in this figure, when the value of capital is around the integer n. The hopping or transition probabilities from one value to another value is shown above the arrows. The entire scenario can as well be depicted via transition matrix $\mathbb{P}_A$ of infinte dimension.

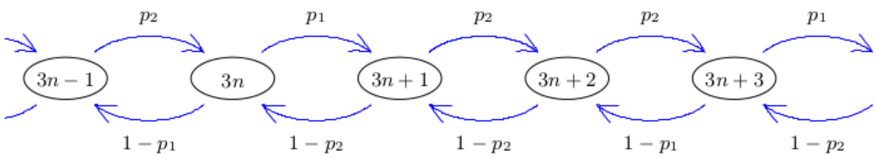

Figure 15: A part of the "doubly-infinite" DTMC is shown in this figure, when the value of capital is around the integer $3n$. The hopping or transition probabilities from one value to another value is shown above the arrows. The entire scenario can as well be depicted via transition matrix $\mathbb{P}_B$ of infinte dimension.

$$\mathbb{P}_A = \begin{pmatrix} 0 & 1-p & \cdots & \cdots & (p) \\ p & 0 & 1-p & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & p & 0 & 1-p \\ (1-p) & \cdots & \cdots & p & 0 \end{pmatrix}$$

The bounds are theoretically infinte but in practice it is a $2N+1$ dimensional matrix for a game played N times. Here the bracketed probabilities are mentioned for the sake of completeness.

4.2 Game B

For the game A, the DTMC is represented in Fig.- 15.

The transition matrix $\mathbb{P}_B$ is given as follows:

$$\mathbb{P}_B = \begin{pmatrix} 0 & 1-p_2 & & & & & (p_2) \\ p_1 & 0 & \ddots & & & & \\ & p_2 & \ddots & 1-p_2 & & & \\ & & \ddots & 0 & 1-p_1 & & \\ & & & p_2 & 0 & 1-p_2 & \\ & & & & p_1 & 0 & \ddots \\ (1-p_1) & & & & & \ddots & \ddots \end{pmatrix}$$

where the losing and winning probabilities in every M_{th} column are $1-p_1$ and p_1 respectively. The bracketed terms are included for completeness.

By extracting the periodic subsystem from the DTMC representation in Fig.15, the dynamics of the games can be more easily studied. The subsystem is be defined by

$$Y_n \equiv X_n mod M$$

Though this representation does not reveal the absolute value of capital, mean ingful trends can be easily calculated. The DTMC defined by Y_n has the states $\{0, \cdots, M-1\}$, and is cyclic. That is, if we win at the highest state $M-1$ we go back to state 0 and vice versa from state 0 to $M-1$. The corresponding DTMC to Y_n is shown in Fig. 16.

For the game A, a module formalism can also be used if we set $p_1 = p_2 = p$. A cyclic DTMC instead of a infinite DTMC helps in analysis of the games. It is similar to that of balancing a chemical reaction using detailed balance. When written from a "back of the envelope" calculation the game B is winning when the clockwise probability is greater than the counterclockwise probability of rotation,i.e

$$p_1 p_2^{M-1} > (1-p_1)(1-p_2)^{M-1}$$

The modulo rule restricts the dimension of transition matrix to $M \times M$.

$$\mathbb{P}_B = \begin{pmatrix} 0 & 1-p_2 & \cdots & \cdots & (p_2) \\ p_1 & 0 & \ddots & \cdots & \cdots \\ \vdots & p_2 & \ddots & 1-p_2 & \vdots \\ \cdots & \cdots & \ddots & 0 & 1-p_2 \\ (1-p_1) & \cdots & \cdots & p_2 & 0 \end{pmatrix}$$

4.3 The *randomized* game

We deal with this randomness by the introduction and use of the mixing parameter γ, which gives the relative probability of choosing the game A at random. Thus the winning probabilities q_1 *and* q_2 are as follows:

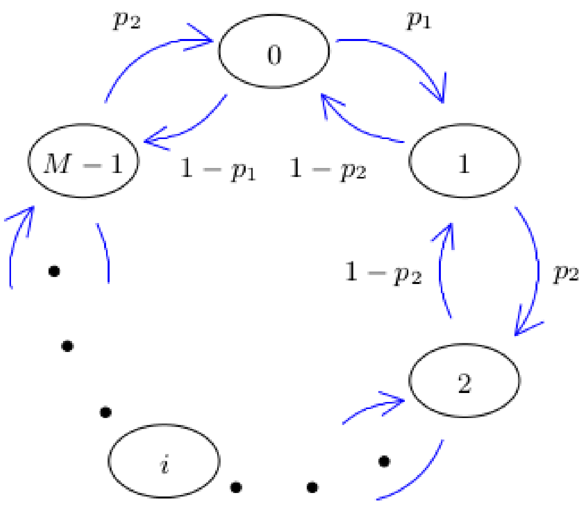

Figure 16: The cyclic DTMC is shown in this figure, when the value of capital is circling around the range $[0, M-1]$. The hopping or transition probabilities from one value to another value is shown above the arrows. The entire scenario can as well be depicted via transition matrix $\mathbb{P}_B$ of $M \times M$ dimension.

$$q_1 = \gamma p + (1-\gamma)p_1, \text{ when capital is a multiple of M}$$
$$and$$
$$q_2 = \gamma p + (1-\gamma)p_2, \text{ otherwise}$$

The corresponding losing probabilities are $1 - q_1$ *and* $1 - q_2$,respectively. The treatment of the randomized game can be done in a similar manner to that of game B as it also forms a DTMC.

4.4 Playing the games analytically

Let, $\pi(n)$ represents the state vector showing the capital after the game is being played n times. Now, if the transition probability between the games is known and given as $\mathbb{P}$, then we have,

$$\pi(n) = \mathbb{P}^n \pi(0)$$

The true distribution of the capital can be attained if $\mathbb{P}$ is constructed keeping the doubly infinite nature in mind. For a trial of N times, it suffices to take $\mathbb{P}$ as a $2N+1$ square matrix. If we start with 0 capital, then $\pi(0) = [\cdots, 0, 1, 0, \cdots]^T$ For the mixing of games via $[a, b]$

$$\pi(n)^{[a,b]} = \mathbb{P}_X^n \pi(0), \text{ where}$$

$$\mathbb{P}_X = \begin{cases} \mathbb{P}_A & \text{if } (n-1) mod(a+b) \ < a, \\ \mathbb{P}_B & \text{otherwise} \end{cases}$$

$$\text{where, } n = 1, 2, \cdots$$

For the mixing of games via γ

$$\pi(n)^{\gamma} = \mathbb{P}_R^n \pi(0)$$
$$\text{where, } \mathbb{P}_R = \gamma \mathbb{P}_A + (1-\gamma)\mathbb{P}_B$$

4.4.1 Calculation of statistical quantities

We define the vector $\mathbf{x} = [-N, \cdots, N]$ which contains all the possible values of the capital (states), when playing N games. Thus the mean(μ_n) and standard deviations(σ_n) are given as

$$\mu_n = \mathbf{x}\pi(n)$$

$$\sigma_n = \sqrt{(\mathbf{x} - \mu_n)^2 \pi(n)}$$

4.5 Equilibrium distribution

The stationary state is reached when,

$$\pi(n+1) = \mathbb{P}\pi(n)$$

Thus at the stationary state we have,

$$\lim_{n \to \infty} \pi(n) = \pi$$

In order to find the states, we need to solve the equation

$$(\mathbb{I} - \mathbb{P})\pi = 0$$

One can take the steady state distribution such as that it is proportional to the diagonal cofactors of $\mathbb{I} - \mathbb{P}$,i.e

$$\pi = \frac{1}{D} \text{diag}(\text{cofac}(\mathbb{I} - \mathbb{P})),$$

where D is the normalization constant. The function 'diag' returns the main diagonal of a matrix and 'cofac' gives the cofactors of a matrix.

For M=3, the states are found, which are given below:

$$\pi^B = \frac{1}{D} \begin{bmatrix} 1 - p_2 + p_2^2 \\ 1 - p_2 + p_1 p_2 \\ 1 - p_1 + p_1 p_2 \end{bmatrix}$$

where, $D = 3 - p_1 - 2p_2 + 2p_1 p_2 + p_2^2$
For game A, $p_1 = p_2 = p$,then

$$\pi^A = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

agreeing with the expected value of having equal probabilities in each state. For game B with no bias($\epsilon = 0$),we have

$$\pi^B = \frac{1}{13}\begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$$

For the randomized game via the paramter γ with no bias($\epsilon = 0$),we have

$$\pi^B = \frac{1}{709}\begin{bmatrix} 245 \\ 180 \\ 284 \end{bmatrix}$$

4.6 Constraints of the games

The motive is find a stationary state for a given set of parameters which make the games A and B losing while the combination a winning one. The winning probabbility is defined as,

$$p_{win} = \sum_{j=0}^{M-1} \pi_j p_j$$

$$\begin{aligned} If\ p_{win} &> 1/2\ then\ \langle X_n \rangle \text{ increases and the game is a winning one.} && (1)\\ &= 1/2\ then\ \langle X_n \rangle \text{ stays steady and the game is a fair one.} && (2)\\ &< 1/2\ then\ \langle X_n \rangle \text{ decreases and the game is a losing one.} && (3) \end{aligned}$$

For the game A, we require,

$$\tfrac{1-p}{p} > 1$$

For the game B, the winning probability becomes,

$$\begin{aligned} p_{win}^B &= \pi_0 p_1 + \pi_1 p_2 + \cdots + \pi_{M-1} p_2 \\ &= \pi_0 p_1 + (1 - \pi_0) p_2 \end{aligned}$$

Putting the proper values of stationary state variables and keeping in mind the condition for B to be losing ($p_{win}^B < 1/2$), we get

$$\tfrac{(1-p_1)(1-p_2)^2}{p_1 p_2^2} > 1$$

For the randomized game to be winning, the condition reads as,

$$\tfrac{(1-q_1)(1-q_2)^2}{q_1 q_2^2} < 1$$

4.7 Range of biasing parameter

The biasing parameter was shown to control whether a game was winning or losing. In terms of Parrondo's paradox, we have shown that randomizing the games improves the performance. However, if is too large then all the games lose, albeit the randomized game does not lose by as much. Conversely if ϵ is too small (negative), then all the games win. Thus,ϵ needs to be chosen such that it biases games A and B to lose, but the improvement gained by mixing is greater than the offset made by ϵ. By substituting the probabilities of the original games into the previous equations, we deduce a range of ϵ for which Parrondo's paradox exist. The equations are respectively,

$$\epsilon > 0$$
$$\epsilon(80\epsilon^2 - 8\epsilon + 49) > 0$$
$$320\epsilon^3 - 16\epsilon^2 + 229\epsilon - 3 < 0$$

For the quadratic part, $b^2 - 4ac < 0$, so the roots are imaginary, meaning that $80\epsilon^2 - 8\epsilon + 49 > 0$ for all ϵ, which leaves $\epsilon > 0$. We can numerically find the roots or use Cardan's method for cubic polynomials to deduce that there is one real and two imaginary roots. Either way the real root is $\epsilon_{max} \sim 0.0131$. which gives the possible range of the biasing parameter as $0 < \epsilon < 0.0131$. To approach the upper limit of this range ϵ_{max} , n needs to be large to offset the initial transient behavior.

5 Examples of Parrondonian effect

The close analogy between the flashing ratchet (continuum model) and Parrondo's games (discrete model) shows that the switching between asymmetry and flat gradient in space can break the drift pattern of a brownian motion, similar to the losing trend being broken by the alterartion between losing game A and asymmetric losing game B. This leads to a logical question of whether this analogy can be extended in other fields or not. What sort of asymmetries can lead to such and effect of "combining two losing combinations to arrive at a winning one", henceforth called as "Parrondonian effect".

5.1 The trueling problem

The truel is similar to a traditional duel except three, rather than two, players have a shoot out. The last man standing is the winner. Here the case of sequential truel, where the gunmen take it in turns to shoot, is considered. The detailed rules are explained with the figure 17, but essentially the weakest Player A has first shot, then Player B, and so on. Intuitively, one may try to eliminate the strongest one out of Player B or Player C as the preferable strategy. The most preferable one, surprisingly, is neither! It turns out that your best strategy for survival is in fact to waste your bullet and shoot into the air.

5.2 The interplay of redundancy and pleiotropy

The term pleiotropy describes an agent that performs multiple tasks, while redundancy is when multiple agents perform the same task. This is clearly illus- trated in figure 18, where we see that pleiotropy can be thought of as the inverse of redundancy. Pleiotropy and redundancy can be ubiquitously seen in many every day networks, ranging from neural interconnections through to client-server based networks made up of server nodes and client nodes.

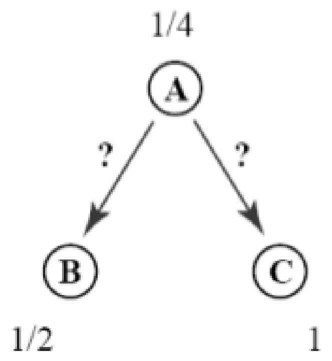

Figure 17: Three gunmen with unlimited amount of ammunition are involved in the game. Each can fire only once in each round of sequential order of firing. The probability of a killshot for them are: $p_A = 1/4,\ p_B = 1/2,\ p_C = 1$, which determines the "strength" or weekness of a player. With the given probabilities, A has the first go, then B, followed by C.

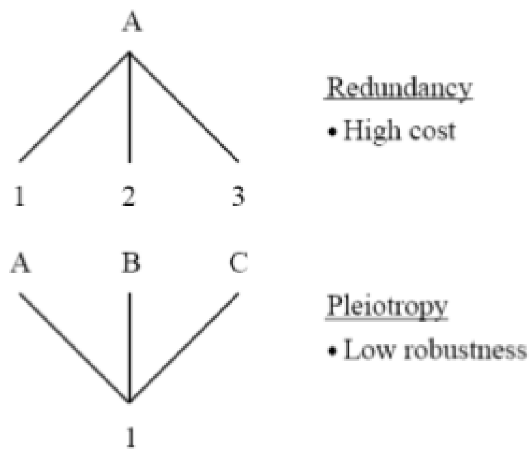

Figure 18:

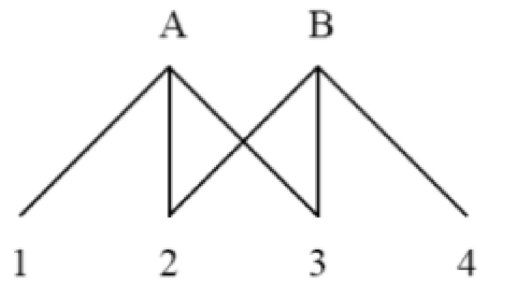

Figure 19:

Figure 18 shows that, individually, pleiotropy and redundancy are rather like 'los- ing games', as redundancy comes at high cost and pleiotropy comes with low robust- ness. Figure 19 illustrates that a mixture or interplay between pleiotropy and redundancy helps to overcome their individual disadvantages. Biological systems provide important examples of pleiotropy and redundancy. Intercellular messenger molecules such as cytokines may act as links between nodes (cells). A deeper knowledge of how pleiotropy and redundancy operate within the cytokine networks, may improve understanding of how to better manipulate disease states [28–30]. To date, little work has been carried out to explore the trade-offs between pleiotropy and redundancy in an evolutionary computational paradigm—future work in this area may help to explore the general principles behind such trade-off in the presence of both limited and unbounded resources. This may enable us to answer a number of fundamental open questions about how real biological, social, and elec- tronic networks are optimally wired.

5.3 Costly signalling

A large area of research where there is a complex interplay of both losing and win- ning strategies is that of "costly signalling". Costly signalling is a term used by evolutionary biologists for the situation whereby an animal advertises its fitness, for example, for procuring a mate. In order to ensure that the signal is 'honest' it has been conjectured that it must come at a cost to the animal—otherwise it would be too easy to send out fake signals. The classic example is the fancy plumage of the male peacock. The larger these feathers are the more attractive the male becomes to his entourage of females. However, the feathers come at cost because (a) they make the male easier to spot by a predator, and (b) the feathers are cumbersome when escaping from a predator. Therefore, the conjecture is that the feathers are an honest signal, because they advertise that the male is fit enough to survive despite them. Thus in order to 'win' and find the optimal mate, the male plays the losing strategy of becoming vulnerable to predators.

5.4 Other examples

Here we will see that an asymmetry in any arbitrary variable can lead to a ratcheting mechanism. The spatial part is well reflected in the Brownian ratchet.

5.4.1 Brazil nut paradox

If we randomly jiggle a bowl of sugar, a bag of flour or a backet of sand, the lumps rise to the top. The scientific name for this phenomenon is the 'Brazil nut paradox', named after that fact that the large Brazil nuts rise to the top when you shake a bag of mixed nuts. Here, the random shaking of the container drives the large nuts 'uphill' against the gravitational gradient and thus this is clearly a Brownian ratchet. The asymmetry in this case lies in the size distribution of the particles and the fact that gravity is directional.

5.4.2 Longshore drift

Another common example is that of longshore drift on a beach. Here, it is common to find that the sand and shells tend to pile up on one end of the beach. This tends to happen when waves come in at an angle to the beachfront. So for example, if we have a south facing beach, and waves impinging in a north-east direction, then sand and shells will tend to pile up on the east side of the beach. Waves will come in a north-easterly direction, but ebb in a southerly direction, drawing out a ratchet-like profile, and dragging material toward the east. Incoming waves loosen the material, reducing frictional forces, and as the waves ebb away friction increases again. Thus the ratchet asymmetry is in the difference between angle of entry and angle of ebb, as well as difference in frictional forces experienced by the material.

When trading on the stock market, a common injunction is to buy-low sell-high in order to ratchet up one's gain. The asymmetry here is in price when we buy and sell, in order to exploit the natural price fluctuations in the market. When paying the restaurant check, at the end of a meal, a client will typically complain if he or she is over charged. However, if the check is accidentally under charged, the client might chose to stay silent. This asymmetry in the transmission of information is used the ratchet up the gain of the client. This is somewhat akin to the previous buy-low sell-high example.

So far we have seen spatial, frictional, informational, and money ratchets—but is a ratchet in the time variable possible? The answer is yes.

5.4.3 Two girlfriend paradox

To illustrate a time ratchet we briefly review the two-girlfriend paradox.

The two-girlfriend problem is a mindboggler that goes as follows. Refering to Fig- 20,the problem stats that Bill arrives at a train station at a random time each day. One train leaves for the east every 10 mins and one train leaves for the west every 10 mins. His strategy is to jump on whichever train arrives first. It turns out on av- erage that he sees Monica nine times more often than Hillary. This seems a little hard to believe given that he arrives at a time random time each day. The answer is that this is a phase (time) ratchet and

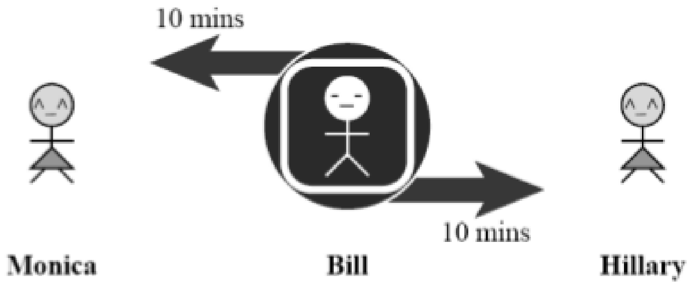

Figure 20: The two girlfriend problem faced by Bill. An asymmetry in time, due to phase difference of east bound and west bound trains and the randomness of Bill's arrival tiem leads to a drift towards Monica from Bill's side.

Scenario	Source of Randomness	Asymmetry
Brazil nut paradox	Shaking the container	Particle sizes/Field
Longshore drift	Waves breaking on the beach	Geometry/Friction
Restaurant check	Waiter's error rate	Information
Buy-low, sell-high	Market fluctuations	Price
2-Girl paradox	Bill's arrival times	Train phase (time)

we must therefore look for an asymmetry in the time variable. In other words, there can be a phase difference between the trains. Imagine a scenario where the eastbound train leaves every 10 mins on the hour, and the westbound train leaves every 10 mins one minute later. If Bill arrives after, say, 10 : 11 am he will have a nine minute window that captures the eastbound train, but if he arrives after 10 : 10 am there is a one minute window in which the westbound train will arrive first. Thus if he arrives randomly, he is more likely to end up in the nine minute window, and thus sees Monica nine times more often.

The follwing table summarizes the a few examples highlighting the different forms of asymmetry we have identified.

5.5 Volatility pumping in stock market

It is a well used technique in stock market. The trick is to mix two types of stocks, one of high risk and high return, The other one is of low risk and low return. Individually, they can be thought of as losing games, providing no net gain in the capital of the investor. But a method of "Portfolio rebalancing" where one can gain in a hefty amount with increasing time. A toy model is explained below:

5.6 Thermodynamics of games of chance

The true analogy between Parrondo's games(discrete version) and Brownian ratchet(continuos version) helps in understanding the discrete-continuum in-

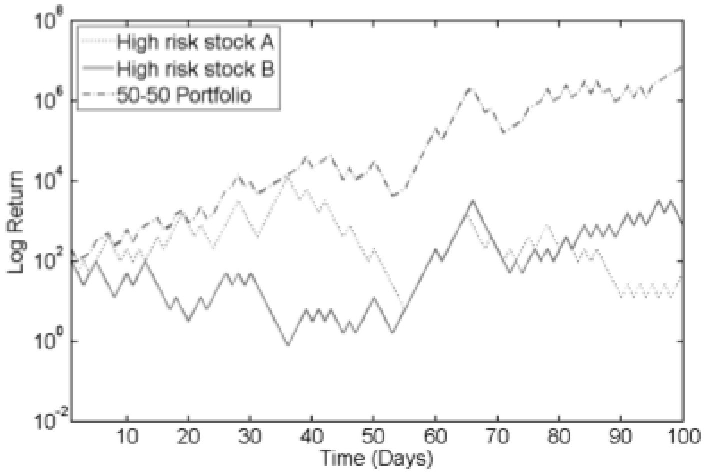

Figure 21: Volatility pumping a low-risk stock with a high-risk stock. The dotted curve simulates a mediocre low-risk stock that in the long run neither wins nor loses. The solid curve represents a volatile stock that gives a 25% expected return, though is high-risk. A simple toy model of volatility is implemented here, where the stock simply halves or doubles, at random, its previous value at each time-step. The chained curve is found by selling both stocks at the end of each time step, adding the total cash to get T, then repurchasing them at the beginning of each time-step at a 50 : 50 split,that is, we purchase $T/2$ worth of the high-risk stock and $T/2$ worth of the low-risk-stock. This is process called *portfolio rebalancing*. Surprisingly, the chained curve grows exponentially, even though the two stocks individually do not perform as well. Both stocks start at Day 1 priced at 100, and thus the combined portfolio (chained curved) starts at 200. The vertical axis is the return in dollars plotted on a logarithmic scale. The return on the rebalanced portfolio is so large that we would not be able to see the individual curves, without the logarithmic plot.

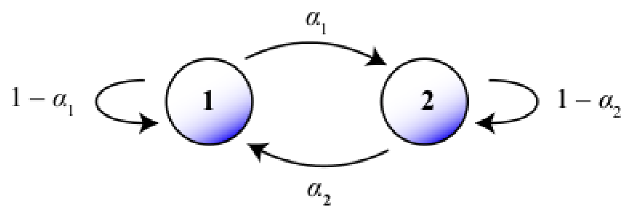

Figure 22: The condition for a non-zero auto correlation between two elements in the final mixed sequence is that $\alpha_1 + \alpha_2 \neq 1$ *and* $\mu_1 \neq \mu_2$, where α_i's are the transition probabilities from state i to state j, and μ_i's are the means of the sequences.

reface for such games of chance. This may lead to development of thermodynamics of all such games in a more rigorous manner. The time reversibility is mapped to unbiased probability and thermodynamic equilibria. While time irreversibility is mapped to the other possibilities in these cases.

5.7 Allison mixture

It is a mixture of two random sequences in random ordering which may lead to an ordered sequence under some restrictions. The state diagrams are explained in the following fig. 22. The process can be related to the sequencing of bases while DNA encoding. Although further progress in this field is yet to be made.

6 Conclusion

6.1 Summary

The fact that and asymmetry and a source of randomness leading to a drift or directional flow is of high interest. The phenomena is even more interesting as it is seen in spatial, temporal, probability space as well in the cases of social dynamics and biological/genetical contexts.

The challenge is to find the portion of phase space where the individual cases are“ losing” and teh mixture is “winning”, in the sense of generating a drift in favoured direction. For Parrondo's original games, it is found and well studied. The biological cases are being taken up as open problems.

The close analogy between Brownian ratchet and Parrondo's games and analytic foundation suggest that it Can serve as a discrete-continuum interface model. Studying this may lead to the development of a general formalism in this field.

6.2 New outlook

The history dependant multiplayer versions of Parrondo's games are still being studied. The latest one being in the present year.

The Allison mixture shows that random mixing of random sequences leads to a correlated or ordered sequence. This concept is thought to be a prospective way of how encoding of DNA takes place.

Quantum mechanical version of the games are being studied. More problems are being looked into where they map onto a physical picture.

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