Dynamics of Eulerian walker

Ayan Bhattacharjee

Seminar Project Presentation, 2014

Why Eulerian walk?

- System and walker affect each other.
- Eulerian walks generate spanning trees.
- Seen in: geological models of river basin distribution, distribution lines of water supply or internet LAN etc.
- Serves as a model of self-organization and self-organized criticality.

System

- A graph G of N nodes
- j^{th} node having τ_j outgoing bonds
- j^{th} node having an outgoing *direction* n_j associated, where $1 \leqslant n_j \leqslant \tau_j \; \forall \; j$
- A Walker is added at any randomly chosen site i
- The configuration is completely specified by the set of values $\{n_i\}$ and the position of the walker

Dynamics

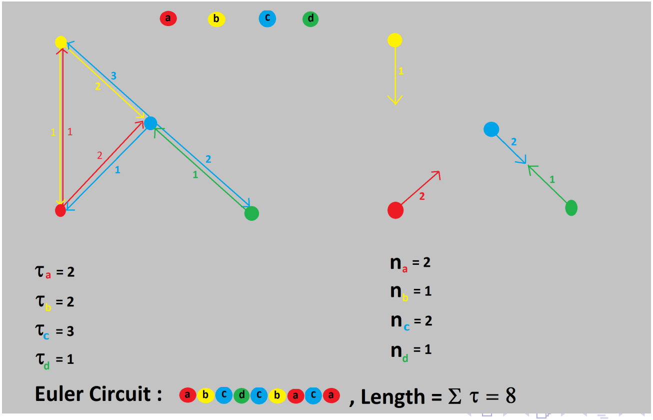
- After arriving at the site j, the walker changes the direction of arrow with the rule:

$$n_j \longrightarrow mod(n_j + 1, \tau_j)$$

- Walker moves along the new direction n_j.
- For an open graph the walker finally leaves the system.
- For a closed system the walker continues to walk forever, finally settling into a *limit cycle* of length $\sum_{i=1}^{i=N} \tau_i$. In this cycle every outgoing bond is visited exactly once. These walks are known as Eulerian circuits.

Eulerian circuit

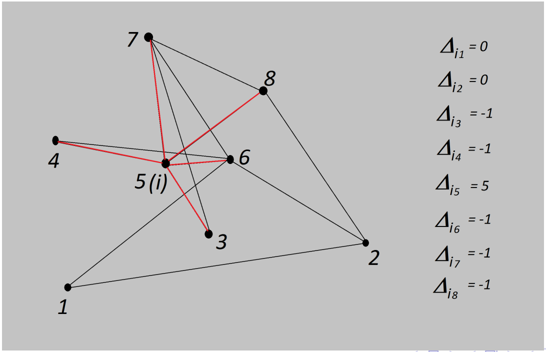
- An Eulerian walk on a closed graph of 4 nodes, where the walker finally settles into the *Eulerian circuit: abcdcbaca* .



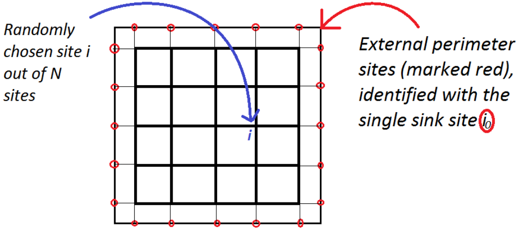
The Δ matrix

For a graph of N points, Δ_{NxN} is defined as:

$$\Delta_{ii} \equiv \text{No of outgoing bonds from site i}$$
$$-\Delta_{ij} \equiv \text{No of bonds from i to j}$$



The particle addition operator a_i

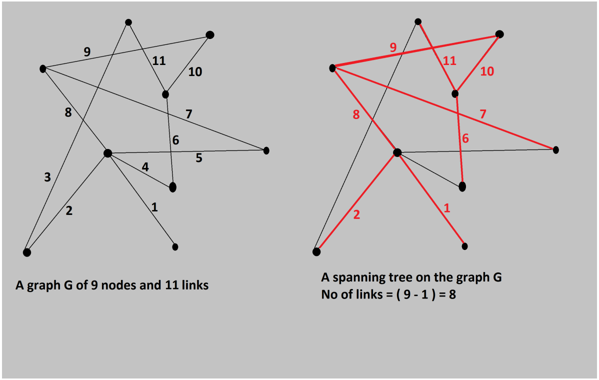


System at configuration C ⇨ Walker added at i ⇨ Walker evolves till reaching i_0 ⇨ System at configuration C′ $C' = a_i C$

- a_i and a_j commute, $i.e\ [a_i, a_j] = 0$
- $\prod_j a_i^{\Delta_{ij}} = I, \forall\ i$

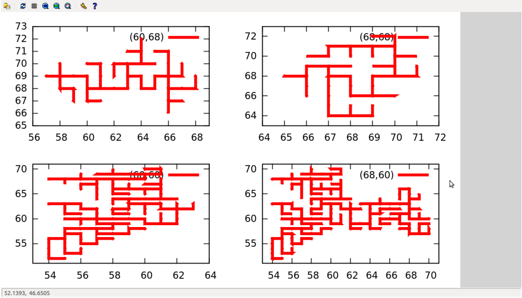
What is a *spanning tree*?

- A collection of (N-1) bonds on a graph of N nodes, which form a single connected cluster.



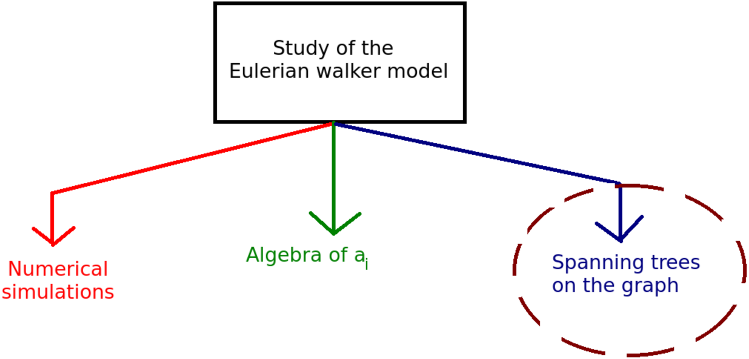
Spanning trees in Eulerian walks

- A Eulerian walk ending at site j, generates a spanning tree rooted at j.
- The figure shows four such spanning trees generated by the "first Eulerian circuit".



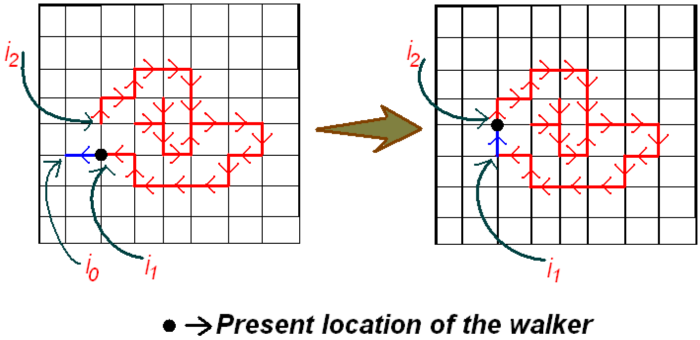
Method of study

- Recurrent states $\Leftrightarrow$ Spanning trees on the graph



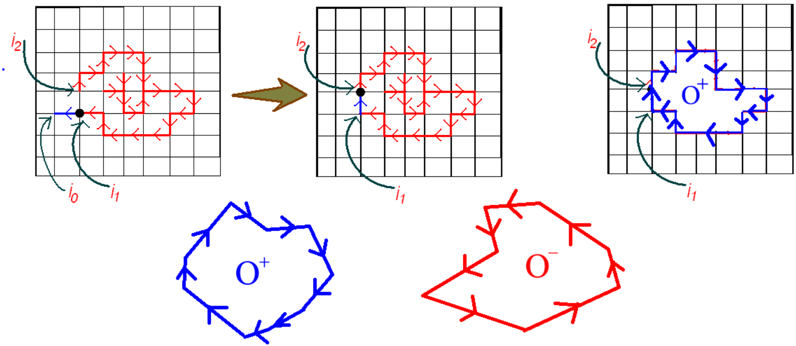
Loop formation on a spanning tree

- The loops are formed when an arrow turns in a manner which encloses a part of spanning tree.



Nature of the loops

- The rule of evolution of arrows determine the nature of the first loop that is formed.



Definition of avalanches

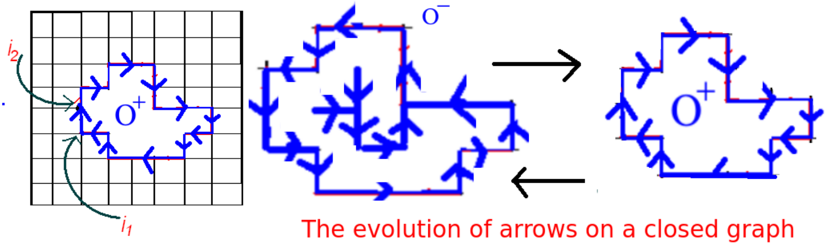
- Recurrent state $\Leftrightarrow$ spanning tree.
  $\Rightarrow$ Particle added.
  $\Rightarrow$ Closing a loop of arrows. $\Leftrightarrow$ loops disappear
- The interval of existance of the loop can be called the avalanche of cyclicity or simply avalanche.

Duration of avalanche

- The time it takes for an Eulerian walker to remain within the first formed loop, is called the duration of avalanche.
- During this time, one loop may transform into another one.
- When the walker leaves the system after all the loops have disappeared from the system, the system settles down to one of its reccurrent states.

Evolution of a loop

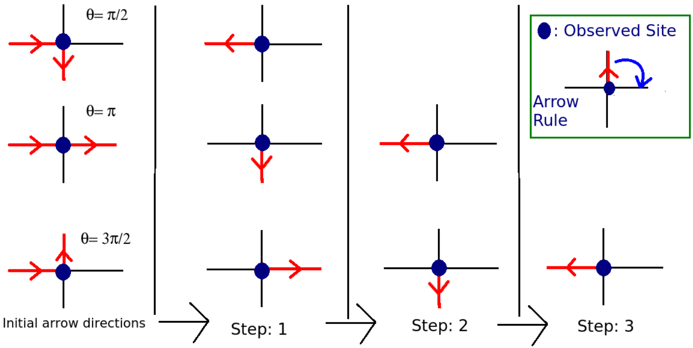
- For a $\mathcal{O}^+$ type graph $\mathcal{L}$, we consider a $\mathcal{O}^-$ type closed graph $\mathcal{G}$.
- The walker stays within the loop until all the arrows on the boundary of the loop are flipped to convert the loop from a $\mathcal{O}^+$ to the loop $\mathcal{O}^-$.



The evolution of arrows on a closed graph

Determining the number of visits

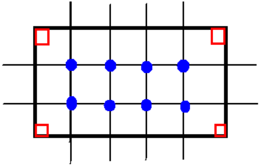
- All inner sites (s) are visited exactly 4 times.
- All perimeter sites, that is associated with an angle $\frac{n\pi}{2}$ are visited exactly n times.



Duration of avalanche T

rectangluar loop

- The simplest loop being a rectangle with a perimeter p which encloses s sites within it.
- Here, total number of steps taken by the walker is identified with the "duration of the avalanche"(T).



No of sites with angle $\pi/2 = 4$

Perimeter: p

No of sites with angle π = (p-4)

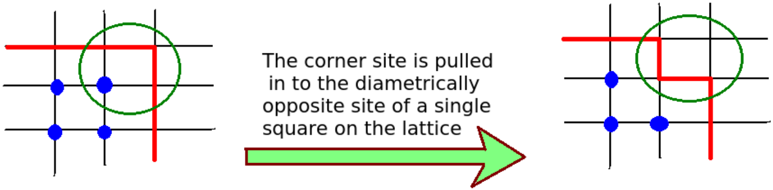
No of inner sites = s

Duration of avalanche

$$T = 4s + 2(p-4) + 4$$
$$= 4s + 2p - 4$$

Duration of avalanche T
arbitrary loops

- Any arbitrary loop on a square lattice can be attained from a rectangular loop of equal perimeter(p), using the follwoing type of transformations, repeatedly.



Duration of avalanche T
arbitrary loops

- Here, number of inner sites go down by 1.
- 2 of the sites with angle π are transformed into sites with angles $\frac{\pi}{2}$.
- A site with angle $\frac{3\pi}{2}$ appears.

$$\begin{aligned} \delta T &= -4-4-1+3+2 \quad (1) \\ &= -4 \quad (2) \end{aligned}$$

- This change in T is already accounted for, as $s \rightarrow (s-1)$, leaving the expression of T unaltered.

Conclusion

- Recurrent states of the Eulerian walker model(EWM) have 1-to-1 correspondance with spanning trees on the same graph. This connection can be exploited to investigate the relaxation process in EWM.
- The formation of loops in the graph leads to avalanche of cyclicity. The nature of loop depends on the rule of arrow-evolution.
- The duration of avalanche can be found out given the perimeter and number of sites within a loop, irrespective of the shape of the loop. In the thermodynamic limit, the duration is connected to the area inside the loop.

References I

- A. M. Povolotsky, V. B. Priezzhev, R. R. Shcherbakov
  Dynamics of Eulerian walkers.
  arXiv: 9802070v1 [cond-mat.stat-mech] , 6 Feb 1998.

- V.B. Priezzhev, D. Dhar, A. Dhar, S. Krishnamurthy.
  Eulerian Walkers as a Model of Self-Organized Criticality.
  Physical Review Letters, 77, 25(1996), 5079.

- S. S. Manna, D. Dhar, S. N. Majumdar
  Spanning trees in two dimensions.
  Physical Review A, 46, 8(1992), R4471.