Self-organized criticality and extreme events:
A Study of BTW Abelian Sandpile Model

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Summer Project Presentation, 2013

Outline

Self-organization:
How Nature Works

- Spontaneous emergence of global correlation from local interactions.The dynamics of the system,itself,guides it from a disordered state to an ordered state.
- Abundant in Nature.

Self-organized criticality ("SOC")

Seen in *driven, dynamic, non-equilibrium* systems

- **Key features:**
  - Critical state: Long ranged spatio-temporal correlation,
  - No external Fine tuning is needed.
- Plausible as a source of natural complexity:
  - Earthquakes
  - Forest Fires
  - Epidemics

What are Xevents?

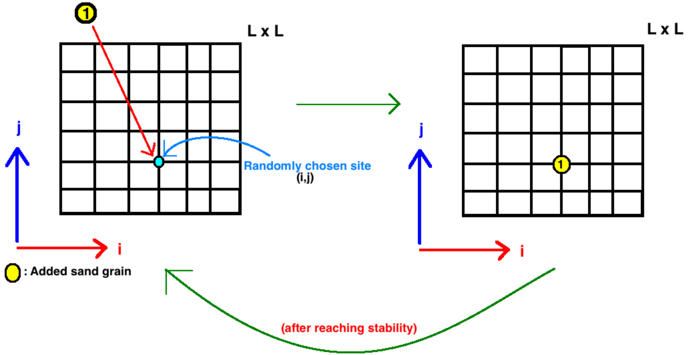
- Events having extreme impacts.
- Occurring in systems with complex dynamics which are usually far from equilibrium.
- Located in the tail end of a probability distribution.
- One of the suggested dynamics is **SOC**

Our study:
The statistics of Xevents in an SOC system

- SOC in a sandpile.
- Temporal correlation among Xevents due to the finiteness of system.
- Predictability of Xevents, depending on externally observed quantities.

The BTW sandpile model:
Description of system

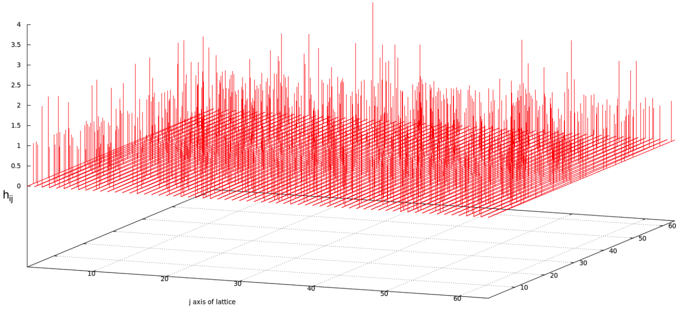
- Addition of sand grain:
  On an LxL square lattice with open boundary condition.



The Process of Addition of Sandgrain

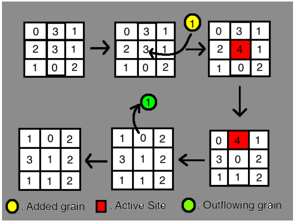
Storage of sand grains:

- In The form of Sand Column:
  Number of grains at (i,j)
  = Height of sand column at (i,j)
  = h_{ij}



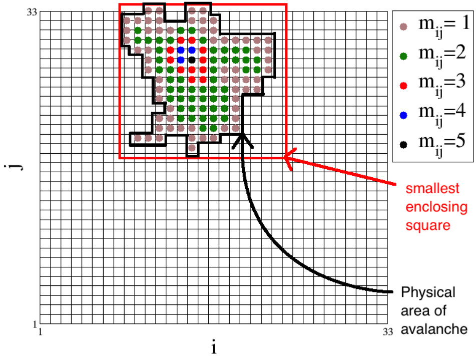
Dynamics of system:
Toppling mechanism and outflow

- Threshold height: $h_c = 4$
- Grain addition: $h_{ij} \longrightarrow h_{ij} + 1$
- If $h_{ij} < h_c$, (**Stable** column)
  $h_{ij} \longrightarrow h_{ij}$
- If $h_{ij} \geq h_c$, (**Unstable** column)
  $h_{ij} \longrightarrow h_{ij} - 4$
  $h_{i'j'} \longrightarrow h_{i'j'} + 1, \; where, (i', j') \equiv (i \pm 1, j), (i, j \pm 1)$



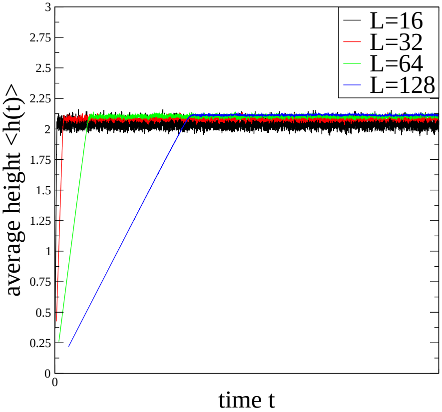
Avalanches:
Meaning and measurement

- **Measurement:**
  - Total number of toppling(S_t)
  - Internal time steps of Avalanche(τ_t)



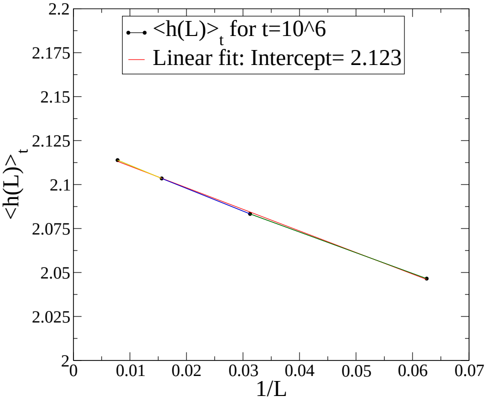
Height of sand column:
Variation with added grains(time)

- Average height of sand column: $\langle h(t) \rangle = \frac{\sum_i \sum_j h(t)_{ij}}{L^2}$



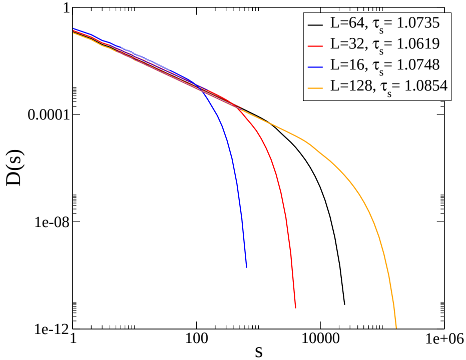
Height of sand column:
Variation with system size

- Time averaged height: $\langle h(L)\rangle_t = \frac{\sum_{t=t_1}^{t=t_2}\langle h(t,L)\rangle}{t_2-t_1}$



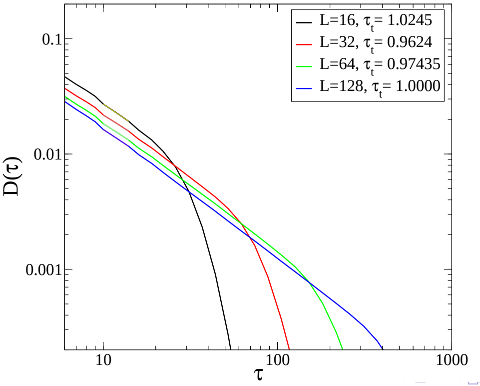
Avalanche dimensions:
Spatial correlation

- If Avalanches of size s_i occurred n_i times out of total N trials: $D(s_i) = \frac{n_i}{N}$



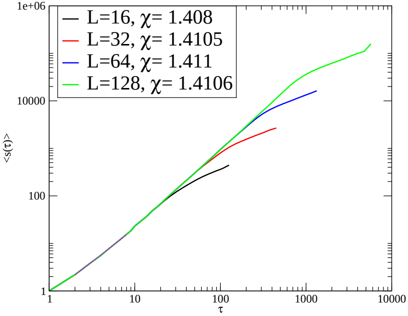
Avalanche dimensions:
Temporal correlation

- If Avalanches of lifetime τ_i occurred n_i times out of total N trials: $D(\tau_i) = \frac{n_i}{N}$



Avalanche dimensions:
Relating lifetime and size

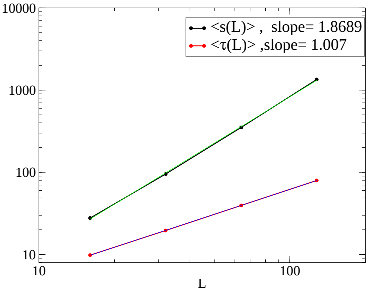
- If Avalanches of lifetime τ had sizes s_i: $\langle s(\tau)\rangle_t = \frac{\sum_{i=1}^{n} s_i}{n}$



Avalanche dimensions

Variation with system size L

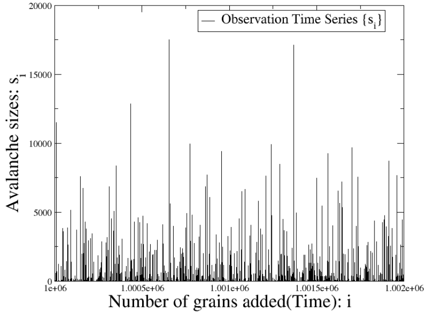
- Average Avalanche Size $\langle s(L)\rangle_t = \frac{\sum_{t=t_1}^{t=t_2} s(t,L)}{t_2 - t_1}$
- Average Avalanche Lifetime $\langle \tau(L)\rangle_t = \frac{\sum_{t=t_1}^{t=t_2} \tau(t,L)}{t_2 - t_1}$



Definition of observation time series:

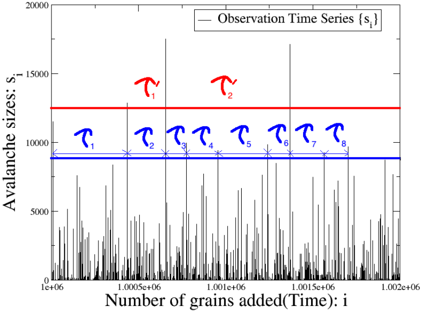
In a sandpile

- The avalanche size,for the addition of i^{th} grain,is denoted by s_i.
- The sequence of $s_i \; \forall \; i \; \in \; (1, N)$ is called the "Observation Time Series"



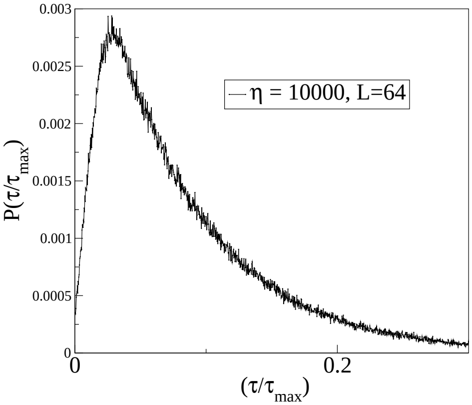
Definition of extreme events:
In a sandpile

- Avalanches having a magnitude greater than a prefixed value η are defined as "**Extreme Events**" in the sand pile.
- Recurrence Time(τ) is the time between two consecutive "extreme" avalanches.



Distribution of recurrence time(τ):
Study of temporal repulsion

- Plot of probability P(τ) vs. normalized recurrence time(τ)



Event series and decision variable:

- Two new quantities are introduced:

$$\begin{aligned} \textit{Event Series } \{X_i\} : X_i &= 1, \text{when } s_i \geq \eta \\ &= 0, \text{when } s_i < \eta \\ \textit{Decision Variable: } y_i &= \sum_{k=1}^{i} a^k s_{i-k}, 0 < a < 1 \\ &= \ ay_{i-1} + as_{i-1} \end{aligned}$$

- y_i takes into account all the past avalanche data but with different weighing factors. The nearer avalanches have more impact and the farther ones have less impact on the value of y_i for certain i.

Conditional probability

- The conditional probability $P(X = 1|y)$ is constructed from the pairs (X_i, y_i). The plot P(X=1|y) vs. y is shown for different η values[2].

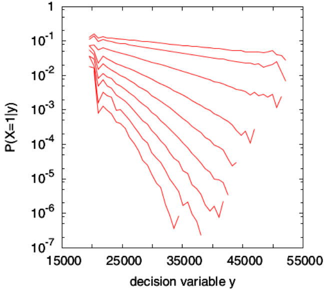
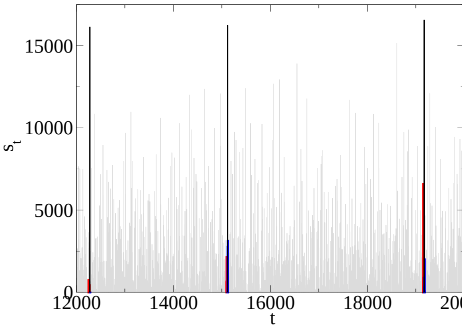


Figure: Figure adapted from A. Garber, S. Hallerberg, H. Kantz, Physical Review E, 80,026124(2009).

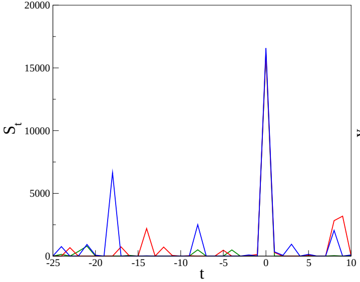
Variation around extreme events:

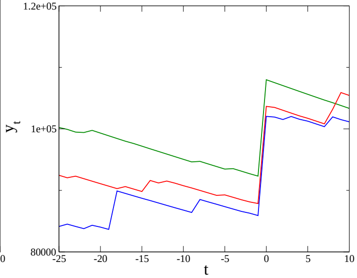
- Snapshot of the observation timeseries showing 3 Xevents(black lines) for $\eta = 16016$. The red lines show 25 previous avalanches and the blue lines mark the subsequent 10 avalanches.



Variation around extreme events:

- Variation of Avalanche size(s_i) and Decision Variable(y_i) around an Extreme Event





Variation around extreme events

- Variation of conditional probability $P(X = 1|y)$ with relative time t,as observed by Garber et al[2].

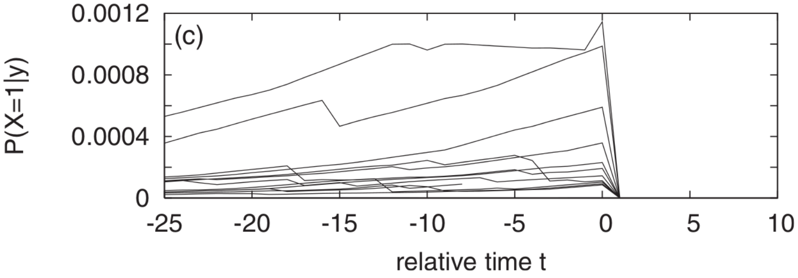


Figure: Figure adapted from A. Garber, S. Hallerberg, H. Kantz, Physical Review E, 80,026124(2009).

Method of prediction

- Deterministic Variable constructed from P(X=1|y):

$$\begin{aligned}\hat{X} &= 1,\ \text{if} P(X = 1|y) > p_c \\ &= 0,\ \text{if otherwise}\end{aligned}$$

- p_c is so chosen:
  - Not too high to make $\hat{X} = 0$ even when extreme events take place.(**Missed events**)
  - Not too low to trigger alarm ($\hat{X} = 1$) for too long a time without any extreme event.(**False alarms**)

Study of predictability:
Receiver operating characteristics

- If hit rate is plotted versus false rate using ROC[2]:

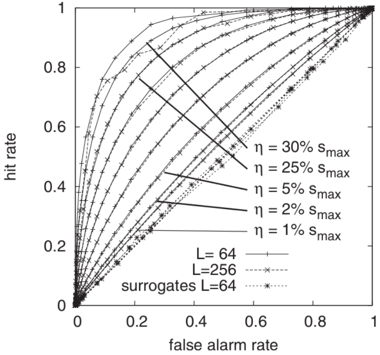


Figure: Figure adapted from A. Garber, S. Hallerberg, H. Kantz, Physical Review E, 80,026124(2009).

Analysis of "ROC" curve:

- Analysis of plot:
  $p_c \to 0 \Rightarrow$ hit rate $\Rightarrow 1,$ False Rate $\Rightarrow 1$
  $p_c \to \; max_y P(X = 1|y) \Rightarrow$ hit rate $\Rightarrow 0,$ False Rate $\Rightarrow 0$
- Diagonal of the plot is the benchmark of prediction. Plots lying above that have better predictability.
- Predictability increases with increasing η.
- If η is expressed as a fraction of $s_{max}(L)$,then Predictability becomes independent of system size L.
- Randomized Surrogate data shows no predictability.

Alternate methods
A comparison

- Average Height ($\langle h \rangle$) and Critical Cluster Size ($s_{critical}$) can be used as Precursors to predict the extreme events.
- Normalized Error Sum: $\epsilon =$Unpredicted Avalanches+Total Alarm Rate
- $\epsilon = [0, 1]$
- The ϵ values for internal parameter dependent predictability and external parameter dependent predictability are almost same.

Conclusion:

- The finiteness of Abelian sandpile leads to a temporal correlation(*repulsion*) between events of large magnitudes.
- This correlation is exploited to predict Xevents in the sandpile,using only the past data of avalanches by Garber et al.

- Future Plans:
  - Further study of extreme events in nature and possible methods of their prediction.

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